Structural estimation of jump-diffusion processes in macroeconomics

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June 2009

Abstract

This paper shows how to solve and estimate a continuous-time dynamic stochastic general equilibrium (DSGE) model with jumps. It also shows that a continuous-time formulation can make it simpler (relative to its discrete-time version) to compute and estimate the deep parameters using the likelihood function when non-linearities and/or non-normalities are considered. We illustrate our approach by solving and estimating the stochastic AK and the neoclassical growth models. Our Monte Carlo experiments demonstrate that non-normalities can be detected for this class of models. Moreover, we provide strong empirical evidence for jumps in aggregate US data.

JEL classification: C13; E32; O40

Keywords: Jump-diffusion estimation; Continuous-time DSGE models; Closed-form

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1 Introduction

This paper shows how to solve and estimate a continuous-time dynamic stochastic general equilibrium (DSGE) model with jumps. It also shows that a continuous-time formulation can make it simpler (relative to its discrete-time version) to compute and estimate the deep parameters using the likelihood function when non-linearities and/or non-normalities are considered. We readopt formulating stochastic models in continuous time (in the tradition of Merton 1975, Eaton 1981, Cox et al. 1985), because for reasonable specifications and parametric restrictions, these models can be solved in closed form.[1] Given the closed-form solution, we compute the likelihood function which then is evaluated for estimation. We illustrate the technique by solving and estimating the stochastic AK and the neoclassical growth models. We highlight two results. First, our Monte Carlo experiments demonstrate that non-normalities can be detected for this class of models. Second, we provide strong empirical evidence of non-normalities in the form of jumps in aggregate US data.

In the field of macroeconometrics, DSGE models have become very successful tools for capturing the main features of business cycle fluctuations. One caveat of the traditional discrete-time formulation is that dynamic general equilibrium models are difficult to solve. It has become popular in macroeconomics to use higher-order approximation schemes to circumvent problems induced by linearizations (Schmitt-Grohé and Uribe 2004). However, most DSGE models in the traditional formulation do not imply a likelihood function that can be evaluated, which makes them even harder to estimate even under small departures from the linear and Gaussian assumptions. Fernández-Villaverde, Rubio-Ramírez and Santos (2006) show that small approximation errors in the solution of the model can have sizable effects on the parameter estimates. Hence the literature is making a huge effort to compute and estimate models with non-linearities and/or non-normalities. These have turned out to be important features of the business cycle for the US economy (Fernández-Villaverde and Rubio-Ramírez 2005, 2007, Justiniano and Primiceri 2008).[2]

Powerful numerical methods have greatly increased the capability of solving complicated DSGE models. Nonetheless, apart from being a pedagogical device, closed-form solutions are complementary because they provide a benchmark for at least three occasions. First, they are the point of reference from which perturbation methods can be used to explore broader classes of models (Judd 1997). Second, they shed light on specific mechanisms, e.g., in which way uncertainty and non-linearities interact (Smith 2007). Finally, as this paper illustrates,

[1]The seminal paper of a stochastic growth model is Brock and Mirman (1972). Recent contributions of continuous-time DSGE models include Corsetti (1997), Wälde (1999, 2005), Steger (2005), Turnovsky and Smith (2006), and Smith (2007). An introduction can be found in Turnovsky (2000).

[2]A review of likelihood-based Bayesian estimation of DSGE models is in An and Schorfheide (2007).

closed-form solutions serve as a benchmark for estimation as they simplify likelihood-based inference especially in the presence of non-linearities and/or non-normalities.

There is a tradition in macroeconomics estimating continuous-time models formulated by systems of stochastic differential equations (among others Phillips 1972, 1991, Hansen and Scheinkman 1995).[3] One challenge when estimating the continuous-time model using discretely sampled observations is that all changes are jumps by construction. In financial econometrics Aït-Sahalia (2004) demonstrates that it is possible to disentangle Brownian noise from jumps even if the jump process exhibits an infinite number of small jumps in any finite-time interval. Our Monte Carlo experiments show that this method can be applied to macroeconomics for reasonable observation frequencies and parameter sets.

This paper is related to the literature on rare disasters (Rietz 1988, Barro 2006), and to the literature where jumps are the determinants of economic growth (cyclical growth models as in Bental and Peled 1996, Matsuyama 1999, Francois and Lloyd-Ellis 2003, Wälde 2005). A key aspect of the empirical analysis is the measurement of the frequency and the sizes of jumps in macroeconomics. We establish that 'rare events' are much more frequent and with smaller jump-sizes than typically assumed for the rare disaster hypothesis.

The remainder of the paper is organized as follows. Section 2 introduces the estimation framework. Section 3 solves continuous-time DSGE models and shows how these models can be estimated. Section 4 conducts the Monte Carlo experiments. Section 5 discusses the empirical results for the US economy. We conclude in Section 6.

2 A framework for likelihood inference

In this section, we describe our framework to perform likelihood-based inference. In order to detect non-normalities in the form of jumps in actual macroeconomic series we employ a similar specification as in Aït-Sahalia (2004). Consider the jump-diffusion process

$$X_\Delta \equiv X_t - X_{t-\Delta} = (\mu - \tfrac{1}{2}\eta^2)\Delta + (B_t - B_{t-\Delta})\eta + \textstyle\int_{t-\Delta}^{t} J_s dN_s, \quad \mu, \eta \in \mathbb{R}. \tag{1}$$

B_t is a standard Brownian motion, J_t is an independent random variable with mean ν and variance γ, and N_t is a standard Poisson process with arrival rate λ.[4] This specification implies, in particular, that the series is independent and identically distributed. It is shown below that many DSGE models indeed imply observables as in (1).

[3]See Gandolfo, ed (1993, chap.1-3) and the references therein for early developments in the field.

[4]It represents the exact solution to the stochastic differential equation $dX_t = (\mu - \frac{1}{2}\eta^2)dt + \eta dB_t + J_t dN_t$. Note that the Poisson process N_t denotes the number of arrivals in a time interval $[0, t)$, and dN_t can either be zero or one. Since B_t is a standard Brownian motion, $B_0 = 0$, $B_{t+\Delta} - B_t \sim \mathcal{N}(0, \Delta)$, $t \in [0, \infty)$.

Since we will be working with maximum likelihood techniques, the jump-size distribution, J_t, has to be fully specified. We assume a binomial distribution. At each instant of time a positive jump in output growth rates (success), $\nu_s \geq 0$, occurs with probability q, whereas a negative jump (disaster), $-\nu_d \leq 0$ occurs with probability $1-q$,[5]

$$J_t = \begin{cases} \nu_s & \text{with} \quad q \\ -\nu_d & \text{with} \quad 1-q \end{cases}. \tag{2}$$

The parameter vector is $\theta \equiv (\nu_s, \nu_d, \lambda, \eta, \mu, q)$, where ν_s and ν_d are the jump terms for positive and negative jumps (ν is the average size of jumps), λ is the arrival rate of the Poisson process, η is the volatility of the Brownian process, μ is the drift of the Brownian process, and q is the probability that a jump is of size ν_s. For the general estimation problem to be well defined, we restrict $(\nu_s, \nu_d, \lambda, \eta) \geq 0$ to be non-negative and $0 \leq q \leq 1$.

2.1 The probability density function

We closely follow Aït-Sahalia (2004) to obtain the probability density function, or transition density of X_Δ, which will be used later for likelihood inference. Conditioned on the events $Q_\Delta = n$ (number of jumps) and $S_\Delta = k$ (number of successful jumps), there must have been exactly n times, say s_i, $i = 1, ..., n$, such that $dN_{s_i} = 1$. Therefore the sum of n independent jump terms is $\int_{t-\Delta}^{t} J_s dN_s = \sum_{i=1}^{n} J_{\tau_i} = \nu_s k - \nu_d(n-k)$. Applying Bayes' rule yields

$$\Pr(X_\Delta \leq x; \theta) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Pr(X_\Delta \leq x | Q_\Delta = n, S_\Delta = k; \theta) \times \Pr(Q_\Delta = n, S_\Delta = k; \theta). \tag{3}$$

Observe that the conditional distribution is

$$\Pr(X_\Delta \leq x | Q_\Delta = n, S_\Delta = k; \theta) = \Phi \left\{ (x - (\mu - \tfrac{1}{2}\eta^2)\Delta - \nu_s k + \nu_d(n-k))/\eta \right\}, \tag{4}$$

where Φ is the Normal cumulative distribution function with mean zero and variance Δ. Moreover, the probability of k successful jumps is

$$\begin{aligned} \Pr(Q_\Delta = n, S_\Delta = k; \theta) &= \Pr(S_\Delta = k | Q_\Delta = n; \theta) \times \Pr(Q_\Delta = n; \theta) \\ &= \frac{\exp(-\lambda\Delta)(\lambda\Delta)^n}{k!(n-k)!} q^k (1-q)^{n-k}. \end{aligned} \tag{5}$$

Inserting both results in (3), we obtain the cumulative distribution function of X_Δ as

$$\Pr(X_\Delta \leq x; \theta) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int_{-\infty}^{(x-\omega)/\eta} \frac{e^{-\frac{1}{2}u^2/\Delta}}{\sqrt{2\pi\Delta}} du \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} q^k (1-q)^{n-k},$$

[5]Note that J_t has mean $\nu = \nu_s q - \nu_d(1-q)$ and variance $\gamma = (\nu_s + \nu_d)^2 q(1-q)$.

where $\omega \equiv (\mu - \frac{1}{2}\eta^2)\Delta + \nu_s k - \nu_d(n-k)$. Applying Leibnitz' rule, it follows that the probability density function of X_Δ is

$$p_X(x, \Delta; \theta) \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \exp\left(-\frac{(x-\omega)^2}{2\Delta\eta^2}\right) \frac{1}{\eta\sqrt{2\pi\Delta}} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} q^k(1-q)^{n-k}. \qquad (6)$$

It shows that the distribution is a *mixture density.* Intuitively, three components are involved, (a) the density of the normal distribution, augmented by (b) terms of the Poisson distribution and (c) terms of the Binomial distribution. Similar formulas used for maximum likelihood estimation are contained in Press (1967), Lo (1988) and Aït-Sahalia (2004).

2.2 Maximum likelihood estimation

Because the process X_Δ in (1) has the Markov property, and because that property carries over to any discrete subsample from the continuous-time path, the average log-likelihood function over N observations has the form

$$\ell_N(\theta) \equiv N^{-1} \textstyle\sum_{i=1}^{N} \ln\{p_X(x, \Delta; \theta)\}, \qquad (7)$$

where θ denotes the parameter vector that maximizes $\ell_N(\theta)$. Owing to difficulties that arise as a result of the complexity of the infinite series representation, the maximum likelihood (ML) approach does not yield explicit estimators for this problem (Press 1967).

We obtain the parameter estimates via numerical evaluation of the log-likelihood in (7). Asymptotic standard errors are based on the outer-product estimate of Fisher's information matrix, $\hat{\mathcal{J}} \equiv N^{-1} \sum_{i=1}^{N} \{h(x, \Delta; \hat{\theta})\}\{h(x, \Delta; \hat{\theta})\}^\top$, for $h(x, \Delta; \hat{\theta}) \equiv \partial \ln\{p_X(x, \Delta; \theta)\}/\partial\theta|_{\theta=\hat{\theta}}$. By the central limit theorem (Hamilton 1994, Aït-Sahalia 2004), the distribution of the ML estimate, $\hat{\theta}$, for sufficiently large sample size N is approximately $\mathcal{N}(\theta_0, \Delta N^{-1}\mathcal{J}^{-1})$.

Using the closed-form density in (6), we can apply standard techniques to test for jumps. Hence, denoting $\hat{\theta}$ as the unrestricted ML estimate, and $\tilde{\theta}$ as the estimate satisfying the restriction $H_0: \; \lambda = 0$, asymptotically we obtain (Sørensen 1991, Hamilton 1994, p.144)

$$2(\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})) \approx \chi^2(r), \qquad (8)$$

where r denotes the effective number of restrictions. In case of four effective restrictions, if the test statistic in (8) exceeds 9.49 (7.78), we can reject the null hypothesis of no jumps ($H_0 : \lambda = 0$) at the 5% (10%) significance level.

2.3 Inferring jumps from large realized observables

In this section, we illustrate how to assign 'probabilities' of occurred jumps to observables. These empirical probabilities could then be used to identify jumps in the data: Given an observation of magnitude x, what is the likelihood that such a change involves a jump?

Applying Bayes' rule to compute the probability of *one* success given the realization x,

$$\begin{aligned}
\Pr(Q_\Delta = 1, S_\Delta = 1 | X_\Delta \geq x; \theta) &= \frac{\Pr(Q_\Delta = 1, S_\Delta = 1, X_\Delta \geq x; \theta)}{\Pr(X_\Delta \geq x; \theta)} \\
&= \frac{(1 - \Pr(X_\Delta \leq x | Q_\Delta = 1, S_\Delta = 1; \theta)) \times \Pr(Q_\Delta = 1, S_\Delta = 1; \theta)}{1 - \Pr(X_\Delta \leq x; \theta)} \\
&= \frac{\left(1 - \Phi\left\{(x - (\mu - \frac{1}{2}\eta^2)\Delta - \nu_s)/\eta\right\}\right) q e^{-\lambda\Delta}\lambda\Delta}{1 - \sum_{n=0}^{\infty}\sum_{k=0}^{n} \Phi\left\{(x - (\mu - \frac{1}{2}\eta^2)\Delta - \nu_s k + \nu_d(n-k))/\eta\right\} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} q^k (1-q)^{n-k}}
\end{aligned}$$

where the probability of *one* success is $\Pr(Q_\Delta = 1, S_\Delta = 1; \theta) = q e^{-\lambda\Delta}\lambda\Delta$. Similarly, one obtains the probability of one disaster, or two or more jumps given the realization x.

3 The macroeconomic theory

Below we solve continuous-time DSGE models (Merton 1975, Eaton 1981, Cox et al. 1985). The main advantage of this formulation is that we can use Itô's formula to easily work with functions of Brownian motions and Poisson processes. Moreover, for many cases we can solve the model by hand and obtain closed-form expressions for the likelihood function.

3.1 The model

Production possibilities. At any time, the economy has some amounts of capital, labor, and knowledge, and these are combined to produce output. The production function is a constant returns to scale technology subject to regularity conditions (see Chang 1988),

$$Y_t = A_t F(K_t, L), \tag{9}$$

where K_t is the aggregate capital stock, L is the constant population size, and A_t is the stock of knowledge or total factor productivity (TFP), which in turn is driven by a standard Brownian motion B_t and a standard Poisson process N_t with arrival rate λ,

$$dA_t = \mu(A_t)dt + \eta(A_t)dB_t + (\exp(J_t) - 1)\, A_{t-} dN_t. \tag{10}$$

In this formulation $\mu(A_t)$ and $\eta(A_t)$ are generic functions satisfying regularity conditions specified below. The jump size is assumed to be proportional to its value an instant before the jump, A_{t-}, ensuring that A_t does not jump negative. The independent random variable J_t has constant mean ν and variance γ and specifies the jump-size distribution.

The capital stock increases if gross investment I_t exceeds capital depreciation, δK_t,

$$dK_t = (I_t - \delta K_t)dt. \tag{11}$$

Preferences. The economy is populated by a large number of infinitely-lived identical individuals, each sufficiently small to neglect effects on aggregate variables. Each individual supplies one unit of labor when labor is productive. The representative consumer maximizes expected life-time utility from the integral of instantaneous utility $u = u(c_t)$ enjoyed from consumption c_t, discounted at the rate of time preference ρ,

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(c_t)dt, \quad u' > 0,\ u'' < 0, \tag{12}$$

subject to

$$da_t = ((r_t - \delta)a_t + w_t - c_t)dt. \tag{13}$$

$a_t \equiv K_t/L$ denotes individual wealth, r_t is the rental rate of capital, and w_t is labor income. The paths of factor rewards are taken as given by the representative consumer.

Equilibrium properties. In equilibrium, factors of production are rewarded with value marginal products, $r_t = Y_K$ and $w_t = Y_L$, and the goods market is cleared $Y_t = C_t + I_t$. Applying Itô's formula (e.g. Protter 2004, Sennewald 2007), the technology in (9) together with (11) and the market clearing condition implies that output evolves according to

$$\begin{aligned} dY_t &= Y_A(dA_t - (\exp(J_t) - 1)\, A_{t-}dN_t) + (Y_t - Y_{t-})\, dN_t + Y_K dK_t \\ &= (Y_A\mu(A_t) + Y_K(Y_t - C_t - \delta K_t))\, dt + Y_A\eta(A_t)dB_t + (\exp(J_t) - 1)\, Y_{t-}dN_t. \end{aligned} \tag{14}$$

It describes a stochastic differential equation (SDE), more precisely a jump-diffusion process which, for solving, demands more information about the behavior of households, $C_t = Lc_t$.

3.2 Obtaining the solution

Solving the model requires the aggregate capital accumulation constraint (11), the goods market equilibrium, equilibrium factor rewards of perfectly competitive firms, and the first-order condition for consumption. It is a system of differential equations determining, given initial conditions, the paths of K_t, Y_t, r_t, w_t and C_t, respectively.

Define the value of the optimal program as

$$V(a_0, A_0) = \max_{\{c_t\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (13) \quad \text{and} \quad (10), \tag{15}$$

denoting the present value of expected utility along the optimal program. It can be shown that the first-order condition for the problem is (cf. Appendix A.1)

$$u'(c_t) = V_a(a_t, A_t) \tag{16}$$

for any $t \in [0, \infty)$. The condition (16) makes consumption a function of the state variables, $c_t = c(a_t, A_t)$, or equivalently, $C_t = C(K_t, A_t)$, and the maximized Bellman equation reads

$$\begin{aligned} \rho V(a_t, A_t) &= u(c(a_t, A_t)) + ((r_t - \delta)a_t + w_t - c(a_t, A_t))V_a + V_A\mu(A_t) + \tfrac{1}{2}V_{AA}\eta(A_t)^2 \\ &\quad + \left(qV(a_t, e^{\nu_s}A_t) + (1-q)V(a_t, e^{-\nu_d}A_t) - V(a_t, A_t)\right)\lambda, \end{aligned} \tag{17}$$

where $r_t = r(a_t, A_t)$ and $w_t = w(a_t, A_t)$ follow from the firm's optimization problem.

The (implicit) solution to the dynamic general equilibrium model is the value function which satisfies both the first-order condition (16) and the maximized Bellman equation (17), subject to appropriate boundary conditions. We use a verification theorem which requires the existence of an optimal control and the existence of a well-behaved indirect utility function for the Bellman equation (Chang 1988, Sennewald 2007). In practice, one makes a guess of the value function and derives the conditions under which this candidate is the solution to the control problem (as shown in Appendices A.2 to A.4). Using the first-order condition, the resulting function $V(a_t, A_t)$ then implies the policy function, $c_t = c(a_t, A_t)$.

3.3 Estimation strategies

In principle, the choice of the model specification should *not* interfere with our objective to detect jumps in macro-data. To see this result use Itô's formula,

$$\begin{aligned} d\ln Y_t &= (\mu(A_t)/A_t + (1 - C(K_t, A_t)/Y_t - \delta K_t/Y_t)Y_K)\, dt - \tfrac{1}{2}\eta(A_t)^2/A_t^2 dt \\ &\quad + \eta(A_t)/A_t dB_t + J_t dN_t. \end{aligned} \tag{18}$$

Notice that the output growth rates per unit of time, $g_\Delta \equiv \ln Y_t - \ln Y_{t-\Delta}$, are obtained by simple integration of (18). Hence the jump term in (1) is independent of specific assumptions about functional forms for $\mu(\cdot)$ and $\eta(\cdot)$ or the policy function $C(K_t, A_t)$. As shown below, the appropriate filtering of transitional dynamics does in fact depend on the models.

In this paper, the state variables are not directly observable, but linked to observables such as aggregate consumption and output. In order to use hypothesis (1) for output growth rates, which gives a closed-form expression for the probability density function, we choose appropriate functional forms for $\mu(\cdot)$ and $\eta(\cdot)$ and employ two estimation strategies. Our first strategy exploits equilibrium conditions to filter model-specific dynamics from observed output growth rates.[6] Then (1) is obtained *exact* by defining

$$g^e_\Delta \;\equiv\; g_\Delta - \textstyle\int_{t-\Delta}^t (1 - C(K_s, A_s)/Y_s - \delta K_s/Y_s) Y_K ds. \tag{19}$$

The correct filter depends on the underlying model and works only for specific cases. Our second strategy neglects actual dynamics in output growth rates and exploits equilibrium moment conditions for the mean. In that case (1) is obtained *approximate*,

$$g^a_\Delta \;\equiv\; E\left(g_\Delta - \textstyle\int_{t-\Delta}^t (\eta(A_s)/A_s dB_s + J_s dN_s)\right) + \textstyle\int_{t-\Delta}^t (\eta(A_s)/A_s dB_s + J_s dN_s), \tag{20}$$

and $E(\cdot) \equiv \lim_{t\to\infty} E_0 \left((1 - C(K_t, A_t)/Y_t - \delta K_t/Y_t)Y_K + \mu(\cdot)/A_t - \frac{1}{2}\eta(\cdot)^2/A_t^2\right) \in \mathbb{R}$ denotes the mean of some limiting distribution. It requires the existence and a correct specification of $E(\cdot)$, but gives reasonable estimates if variables are close to the mean.[7] Section 4 provides Monte Carlo evidence suggesting that both strategies work in practice.

Thus the explicit solutions below are used *(a)* to provide proofs of existence and their necessary conditions, *(b)* to study the implications of specific functional forms, and *(c)* to obtain a representation as in (1) that accounts for model-specific characteristics.

3.4 Explicit solutions

We restrict our attention to the widely used class of utility functions which is characterized by constant intertemporal elasticity of substitution (CES),

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \tag{21}$$

where $\sigma = 1$ refers to $u(c_t) = \ln c_t$ (neglecting a constant for notational convenience).

[6]If the assumed TFP process for (10) has the Markov property, our 'filtered' output growth rates will be Solow residuals, $\ln A_t - \ln A_{t-\Delta}$. Hence, as an alternative approach, the estimation method can be used as a reduced form analysis for establishing jumps in macroeconomics using Solow residuals.

[7]This approach is different from approximating the continuous-time representation with a discrete-time model which takes into account the dynamics of the system (see e.g. Gandolfo, ed 1993, p.20). It neglects model-specific dynamics and assumes that observables are drawn directly from the limiting distribution.

3.4.1 The stochastic AK model

Specification. Suppose that output is generated according to a linear technology,

$$Y_t = A_t K_t, \tag{22}$$

which means that all input factors can be accumulated (labor is not a factor of production).[8] As a result, the rental rate of capital, $r_t = Y_K = A_t$, replicates the dynamics of TFP,

$$dr_t = \mu(r_t)dt + \eta(r_t)dB_t + (\exp(J_t) - 1)\, r_{t-}dN_t. \tag{23}$$

Since capital is the only accumulable factor, its rewards can be interpreted as a very short run (certain) interest rate (Merton 1975). We use this result to make plausible assumptions on $\mu(\cdot)$ and $\eta(\cdot)$ linking the model's equilibrium capital rewards to models of the short-rate process in other areas of research (cf. Aït-Sahalia 1996).

Proposition 3.1 (Log-preferences) *Given any process in (10) that implies boundedness of life-time utility (12) for all admissible consumption paths, if elasticity of intertemporal substitution (EIS) is* $1/\sigma = 1$*, then optimal consumption is linear in wealth.*

$$1/\sigma = 1 \quad \Rightarrow \quad c_t = c(a_t) = \rho a_t \tag{24}$$

Use the policy function (24), $C_t = Lc_t = \rho K_t$, and the resource constraint (11) to obtain the growth rate of aggregate consumption $g^c_\Delta \equiv \ln C_t - \ln C_{t-\Delta} = -(\rho + \delta)\Delta + \int_{t-\Delta}^t r_s ds$.

Case 3.2 (Standard TFP) *Suppose (10) is geometric, i.e.,* $\mu(A_t) = \mu A_t$, $\eta(A_t) = \eta A_t$,

$$dr_t = \mu r_t dt + \eta r_t dB_t + (\exp(J_t) - 1)\, r_{t-}dN_t, \quad \mu, \eta \in \mathbb{R}. \tag{25}$$

Employing the dynamics for standard TFP in (18), and inserting $C_t = \rho K_t$ yields

$$g_\Delta \;=\; -(\rho + \delta)\,\Delta + \int_{t-\Delta}^t r_s ds + (\mu - \tfrac{1}{2}\eta^2)\Delta + (B_t - B_{t-\Delta})\eta + \int_{t-\Delta}^t J_s dN_s. \tag{26}$$

Discussion. The dynamics imply Merton's (1973) interest rate process in (25), allowing for jumps. An important caveat is that r_t, in general, is not stationary, hence does not have a limiting distribution. It implies a parameter restriction such that $E_0(A_t) \neq 0$ exists for large t and pins down the admissible parameter set to $\mu = (1 - e^{v_s} q - e^{-\nu_d}(1-q))\lambda$, and the boundedness condition reduces to $\rho > 0$. This restriction can be tested empirically.

[8] It describes an economy similar to deterministic models in Romer (1986), Jones and Manuelli (1990), and Rebelo (1991). A slightly different approach is taken in Eaton (1981) (for a discussion see Smith 2007).

Estimation. Observe that our *exact* estimation strategy can be applied as follows. From (26) we use $g^e_\Delta = g_\Delta - g^c_\Delta$ to obtain filtered growth rates (19). Because of $E_s(A_t) = A_s$ is not unique $\forall s$, the *approximate* approach cannot be applied here.

Case 3.3 (Stationary TFP) *Suppose (10) is geometric-reverting, $\mu(A_t) = c_1 A_t(c_2 - A_t)$, $\eta(A_t) = \eta A_t$, and let $c_2 \equiv \rho + \delta + \mu/c_1$ denote the non-stochastic steady state,*

$$dr_t = c_1 r_t(c_2 - r_t)dt + \eta r_t dB_t + (\exp(J_t) - 1)\, r_{t-} dN_t, \quad \mu, \eta, c_1, c_2 \in \mathbb{R}_+. \tag{27}$$

Employing the dynamics for stationary TFP in (18), and inserting $C_t = \rho K_t$ yields

$$g_\Delta = (1 - c_1)\big(- (\rho + \delta)\Delta + \textstyle\int_{t-\Delta}^t r_s ds\big) + (\mu - \frac{1}{2}\eta^2)\Delta + (B_t - B_{t-\Delta})\eta + \textstyle\int_{t-\Delta}^t J_t dN_t. \tag{28}$$

Discussion. The dynamics imply Constantinides' (1992) reverting interest rate process in (27) accounting for jumps. In that c_1 is the speed of reversion and c_2 is the non-stochastic steady state. Moreover, r_t has a limiting distribution where $E(r) = (c_1 c_2 - \frac{1}{2}\eta^2 + \nu\lambda)/c_1$, the boundedness condition is $\rho > \mu + (e^{\upsilon_s} q + e^{-\nu_d}(1 - q) - 1)\lambda$. Observe that for $c_1 = 1$ output growth rates do not exhibit patterns of autocorrelation, that means consumption dynamics exactly offsets the effects of reverting capital rewards (27).

Estimation. The *exact* approach uses $g^e_\Delta = g_\Delta - (1 - c_1) g^c_\Delta$ to get filtered growth rates (19), but requires knowledge of the speed of reversion ($c_1 \neq 1$). The *approximate* strategy employs $g^a_\Delta = (\mu - \frac{1}{2}\eta^2)/c_1\Delta + \frac{1-c_1}{c_1}\nu\lambda\Delta + (B_t - B_{t-\Delta})\eta + \int_{t-\Delta}^t J_s dN_s$ with the same caveat. It gives $E(g_\Delta) = (\mu - \frac{1}{2}\eta^2 + \nu\lambda)/c_1\Delta$ as the mean of some limiting distribution.

3.4.2 The stochastic neoclassical model

Specification. Suppose that production takes the form (Mirrlees 1974, Merton 1975)

$$Y_t = A_t K_t^\alpha L^{1-\alpha}, \tag{29}$$

which means that only physical capital can be accumulated, labor is productive, and let

$$dA_t = \mu A_t dt + \eta A_t dB_t + (\exp(J_t) - 1)\, A_{t-} dN_t, \quad \mu, \eta \in \mathbb{R}. \tag{30}$$

It seems reasonable to use standard (non-stationary) dynamics for the TFP process in (10), because for the neoclassical model A_t is the only source of long-run economic growth. As an immediate result, the rental rate of capital, $r_t = Y_K = \alpha A_t K_t^{\alpha-1} L$, obeys

$$dr_t = \tfrac{1-\alpha}{\alpha}\left(\alpha C_t/K_t + \alpha\delta + \mu\alpha/(1-\alpha) - r_t\right) r_t dt + \eta r_t dB_t + (\exp(J_t) - 1)\, r_{t-} dN_t, \tag{31}$$

which depends on households optimal consumption decisions. Due to decreasing returns to scale, the rental rate of capital in itself is a reverting process (see also Merton 1975). It is shown below that we obtain the same dynamics as in (27) for two cases.

Proposition 3.4 (Linear-policy-function) *Given any process in (10) that implies boundedness of life-time utility (12) for all admissible consumption paths, if the output elasticity of capital is the reciprocal of the EIS, then optimal consumption is linear in wealth.*[9]

$$\alpha = \sigma \quad \Rightarrow \quad c_t = \phi a_t \quad \textit{where} \quad \phi \equiv (\rho + (1-\sigma)\delta)/\sigma \tag{32}$$

Use the policy function (32), $C_t = Lc_t = \phi K_t$, and the resource constraint (11) to obtain the growth rate of aggregate consumption $g^c_\Delta \equiv \ln C_t - \ln C_{t-\Delta} = -(\phi+\delta)\Delta + 1/\alpha \int_{t-\Delta}^t r_s ds$. Employing the dynamics for standard TFP in (18), and inserting $C_t = \phi K_t$ yields

$$g_\Delta = -\alpha(\phi+\delta)\Delta + \int_{t-\Delta}^t r_s ds + (\mu - \tfrac{1}{2}\eta^2)\Delta + (B_t - B_{t-\Delta})\eta + \int_{t-\Delta}^t J_s dN_s. \tag{33}$$

Discussion. It is notable that (32) endogenously implies Constantinides' (1992) reverting interest rate process in (31) allowing for jumps. Defining parameters for the speed of reversion $c_1 \equiv \frac{1-\alpha}{\alpha}$ and the non-stochastic steady state $c_2 \equiv \rho+\delta+\mu/c_1$, then r_t has the same dynamics as in (27), thus a limiting distribution with $E(r) = (c_1 c_2 - \frac{1}{2}\eta^2 + \nu\lambda)/c_1$. The boundedness condition is $\rho > \mu + \lambda(e^{v_s}q + e^{-\nu_d}(1-q) - 1)$ (as shown in Appendix A.5).

Estimation. Because of established priors about the output elasticity of capital α, the *exact* approach may use $g^e_\Delta = g_\Delta - \alpha g^c_\Delta$ to get filtered growth rates (19). The *approximate* strategy employs $g^a_\Delta = \frac{1}{1-\alpha}(\mu - \frac{1}{2}\eta^2)\Delta + \frac{\alpha}{1-\alpha}\nu\lambda\Delta + (B_t - B_{t-\Delta})\eta + \int_{t-\Delta}^t J_s dN_s$ and gives $E(g_\Delta) = \frac{1}{1-\alpha}(\mu - \frac{1}{2}\eta^2 + \nu\lambda)\Delta$ as the mean of some limiting distribution.

Proposition 3.5 (Constant-saving-function) *Given any process in (30) that implies boundedness of (12) for all admissible consumption paths, if the subjective discount factor is* $\bar\rho \equiv (\alpha\sigma - 1)\delta - \sigma\mu + \frac{1}{2}\sigma(1+\sigma)\eta^2 + (e^{-\sigma\nu_s}q + e^{\sigma\nu_d}(1-q) - 1)\lambda$, *then optimal consumption is a constant fraction of income.*

$$\rho = \bar\rho \quad \Rightarrow \quad c_t = (1-s)A_t a_t^\alpha, \quad \sigma > 1, \quad \textit{where} \quad s \equiv 1/\sigma \tag{34}$$

Use the policy function (34), $C_t = Lc_t = (1-s)Y_t$, and the SDE for output (14) to obtain the immediate result that consumption simply replicates the dynamics of output. Employing

[9] This parametric restriction is fairly well established in macroeconomics (Chang 1988, Xie 1991, 1994, Boucekkine and Tamarit 2004, Wälde 2005, Smith 2006, 2007).

the dynamics for standard TFP in (18), and inserting $C_t = (1-s)Y_t$ yields

$$g_\Delta = -\alpha\delta\Delta + 1/\sigma \int_{t-\Delta}^{t} r_s ds + \mu\Delta - \tfrac{1}{2}\eta^2\Delta + (B_t - B_{t-\Delta})\eta + \int_{t-\Delta}^{t} J_s dN_s. \tag{35}$$

Discussion. Similar to the linear-policy function (32), the solution in (34) endogenously implies Constantinides' (1992) reverting interest rate process in (31) accounting for jumps. We obtain the speed of reversion $c_1 \equiv \frac{1-\alpha}{\alpha\sigma}$, the non-stochastic steady state $c_2 \equiv \alpha\sigma\delta + \mu/c_1$, and the mean of the limiting distribution $E(r) = (c_1 c_2 - \frac{1}{2}\eta^2 + \nu\lambda)/c_1$. Because consumption has the same dynamics as output, there is no way to filter output growth rates for model dynamics. It can be shown that $\rho > (\mu - \frac{1}{2}\eta^2 + \nu\lambda)(1-\sigma)/(1-\alpha)$ is sufficient for boundedness.

Estimation. There is no way to employ the *exact* approach because the solution has the (counterfactual) implication that output and consumption growth rates are identical. The *approximate* approach employs $g_\Delta^a = \frac{1}{1-\alpha}\left(\mu - \frac{1}{2}\eta^2\right)\Delta + \frac{\alpha}{1-\alpha}\nu\lambda\Delta + \eta(B_t - B_{t-\Delta}) + \int_{t-\Delta}^{t} J_s dN_s$, which is the same as for any solution to the neoclassical growth model. Intuitively, this results is obtained because by assumption growth is exogenous, i.e., the mean growth rate of some non-degenerated limiting distribution will not depend on household's behavior.

To summarize, we obtain two solutions to the neoclassical model and we can thus address the approximation error when using the *approximate* approach for plausible scenarios.

4 Monte Carlo evidence

A legitimate question is whether the estimation methods, usually applied for high frequency data, have relevance at the discrete observation frequencies of macro-observables. As shown in Aït-Sahalia (2004), increasing the frequency of observations reduces the Brownian noise holding the jump-size constant (time-smoothing effect). In most applications, the higher is the observation frequency, the higher is the probability that a jump can be recognized as such from large realizations. Macroeconomists, however, can make use of monthly or quarterly data only. A simulation experiment estimating the parameters of a continuous-time process using discrete observations at these frequencies seems important.

Another issue is how much model-specific dynamics complicate the estimation of deep structural parameters. We address this question by comparing results of both the *exact* and the *approximative* estimation strategy to obtain the size and effects of the approximation error for the neoclassical DSGE model. Note that for the exact approach (model *I*) the choice of the model is irrelevant in the Monte Carlo experiments (all variables are observed), whereas for the approximate approach the crucial parameter is the speed of reversion of capital rewards. We use two scenarios, one has lower speed of reversion, $c_1 = .5$ (model *II*), and the other has relatively high speed of reversion, $c_1 = 3$ (model *IIa*).

Starting with the stochastic AK model in (28), we simulate $M = 5000$ sample paths each of length $N = 580$ at frequency $\Delta = 1/10$. In the figures below we report the results for $(\nu_s, \nu_d, \lambda, \eta, \mu, q) = (.025, .02, .8, .02, .01, .5)$, and set other parameters at realistic values $(\rho, \alpha, \sigma, \delta) = (.03, .5, 1, .05)$ for comparability with the neoclassical model. The number of observations of each series and the parameters were chosen such that it roughly coincides with available (monthly) aggregate US data. Given realized stochastic processes we then obtain sample parameters $(\nu_s, \nu_d, \lambda^s, \eta^s, \mu^s, q^s)$ for comparison with the estimates. Thereafter we study the implications of sampling at quarterly frequency, $\Delta = 1/4$, because often aggregate data is not available at the monthly basis.

The results are summarized in Table 4 and Figures 1 and 3. In our experiments, the likelihood-ratio test (8) rejects the null hypothesis of no jumps in 100% at the 1% significance level. For comparison, simulating from the model without jumps, the likelihood-ratio test rejects the null in only 2.4% at the 5% significance level. All parameter estimates remain in a small interval around the sample values despite the fact that the data are sampled at monthly frequency ($\Delta = 1/10$). As one caveat, the arrival rate is on average estimated too high, $1/M \sum_i \hat{\lambda}_i = 0.8971 > 1/M \sum_i \lambda_i^s = .7905$. This phenomenon can be attributed to the well-known identification problem that may arise if the sample size is small or moderate (cf. Aït-Sahalia 2004). From Figure 1, the histogram of ML estimates is skewed right for the arrival rate, $\hat{\lambda}$, and skewed left for the Brownian noise, $\hat{\eta}$. Occasionally the jumps cannot be correctly disentangled from the Brownian noise. In such cases we obtain a too high estimate for the arrival rate, tiny jump-sizes, and the estimate of the Brownian noise is too small. Fortunately such problems of identification do not arise very often in practice. For illustration, considering the .95 quantile of the estimates for $\hat{\lambda}$, this small sample bias and the dispersion of $\hat{\lambda}$ is reduced substantially (cf. Table 5).[10]

Another way of accommodating the identification problem is by fixing jump-sizes to plausible values. Then, the problem of identification disappears and parameter estimates are unbiased for correct values ν_s and ν_d (cf. models *Ia* and *IIb* in Table 4 and Figure 4). Hence, fixing the size of jumps, or equivalently, giving *a priori* information, does not only yield better finite sample properties of estimators in the sense of lower dispersion, but it also removes identification problems that arise due to small sample sizes.

Comparing the *exact* and the *approximate* estimation strategy, we find small effects of the approximation error on parameter estimates. As in Table 4 and Figures 2 and 3, the most sensitive measures to approximations and therefore to different speed of reversion are the dispersion of the estimates on the arrival rate $\hat{\lambda}$ and the drift parameter $\hat{\mu}$. Other estimates

[10]Often, problems of identification can be detected in practice from an insignificant parameter estimate along with an implausible high arrival rates and tiny jump sizes.

are virtually unaffected by the different values of the speed of reversion. It is interesting to note that for speed of reversion $c_1 = .5$ the approximate approach even slightly outperforms the exact approach (compare models *I* and *II*). It seems that the approximation error partly counteracts the time-smoothing effect by 'adding' some persistence to the data. Thus we can safely conclude that the equilibrium dynamics of our models do not necessarily complicate the estimation of the structural parameters.

The results for $\Delta = 1/4$ are in Tables 6 and 7 and Figures 5 and 6. The power of the test in (8) reduces to 76.1% at the 5% significance level, while in the model without jumps the null is (falsely) rejected in 4.5% at the same level of significance. As the histograms illustrate, the dispersion increases substantially and the time-smoothing effect becomes severe. It seems necessary to accommodate the identification problem by providing *a priori* information, e.g., fixing the jump-sizes (models *Ia* and *IIb*). Then parameter estimates remain in a reasonable interval around sample values given that the data is sampled at a quarterly basis.

Summarizing the Monte Carlo experiments, jumps in macro-time series can be detected. If the data were sampled at a monthly basis, the parameter estimates can simply be obtained by maximizing the likelihood function implied by our hypothesis (1). It even works for data at quarterly frequency when sufficient *a priori* information is provided. The general equilibrium dynamics do not severely complicate the estimation of deep parameters of interest.

5 Empirical results

This section reports empirical estimates of our DSGE models. We search for non-normalities in the form of jumps in aggregate US data from 1960:Q1 to 2008:Q4. Our sample length is limited by the availability of data on monthly consumption. We employ the monthly index of industrial production of major manufacturing industries (IP) from the Federal Reserve statistical release (G.17), and NIPA data on real gross domestic product and consumption expenditures from the Bureau of Economic Analysis (BEA), all seasonal adjusted.

Table 1 reports the MLE for the 6 parameters of the model for quarterly real GDP. Their asymptotic standard errors and the likelihood-ratio test (8) are in parentheses. We obtain similar parameter estimates for different DSGE models. The *exact* approach gives estimates of the stochastic AK model with standard TFP (model *I*), the stochastic AK model with stationary TFP for $c_1 = .5$ (model *Ia*) and $c_1 = .7$ (model *Ib*), the stochastic neoclassical model with a linear-policy function for $\alpha = .5$ (model *Ia*) and $\alpha = .3$ (model *Ib*). Similarly, the *approximate* approach gives the estimates of the stochastic AK model with stationary TFP for $c_1 = .5$, and the stochastic neoclassical model for $\alpha = .5$ (both model *II*). Different calibrations of c_1 or α only affect the MLE of the drift component μ (not shown).

Table 1: US quarterly real GDP from 1960:Q1 to 2008:Q3

Model		*Parameter estimates*						*Value*
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$	$\ell_N(\theta)$
eq. (19)	*I*	0.0102	0.0109	1.0869	0.0079	0.0019	0.3450	706.7
		(0.0013)	(0.0009)	(0.2012)	(0.0005)	(0.0011)	(0.0733)	(14.63)
	Ia	0.0071	0.0095	2.5570	0.0051	0.0124	0.6343	705.5
		(0.0003)	(0.0003)	(0.2224)	(0.0003)	(0.0008)	(0.0285)	(17.98)
	Ib	0.0084	0.0095	2.2307	0.0063	0.0192	0.5934	689.7
		(0.0006)	(0.0005)	(0.2569)	(0.0004)	(0.0010)	(0.0378)	(13.63)
eq. (20)	*II*	0.0111	0.0141	1.4340	0.0088	0.0121	0.6688	663.3
		(0.0009)	(0.0009)	(0.2111)	(0.0005)	(0.0013)	(0.0468)	(15.68)
	IIa	0.0150	0.0200	0.4313	0.0129	0.0166	0.5339	659.9
				(0.0939)	(0.0006)	(0.0012)	(0.0937)	(8.88)

Notes: This table reports the ML estimates for the following model specifications and calibrations: using the *exact* approach, for the standard AK (model *I*, $\alpha = 1$), the stationary AK (model *Ia* $c_1 = .5$ and *Ib* $c_1 = .7$) and the neoclassical model with a linear-policy function (model *Ia* $\alpha = .5$ and *Ib* $\alpha = .3$). Using the *approximate* approach, it reports the ML estimates for the stationary AK (model *II*, $c_1 = .5$) and the neoclassical model (model *II* $\alpha = .5$), as well as for fixed jump-size (model *IIa*). Standard errors and likelihood-ratio tests (8) are in parentheses ($\Delta = 1/4$; $N = 195$).

As shown in the Monte Carlo experiments, with quarterly data it is difficult to disentangle the jumps from the diffusion. Though the arrival rate seems far too high, the null hypothesis of no jumps is rejected at the 5% significance level for all models (the critical value is 9.49). To accommodate the identification problem (positive bias of $\hat{\lambda}$ together with a negative bias of $\hat{\nu}_s$ and ν_d), we constrain jump-sizes at larger values (model *IIa*), which loosely speaking makes it less likely for an observation to be identified as a jump. It also accounts for the fact that negative jumps are more pronounced on average. Nonetheless the null hypothesis of no jumps is rejected at the 5% significance level (the critical values is 5.99).

Imposing the restriction $\nu_s = .015$ (or $\nu_d = .02$) means that the direct effect of one jump on annual output growth in (18) is an increase (or drop) by 1.5 (or 2) percentage points. The MLE gives the arrivals of such rare events every $1/\hat{\lambda} = 2.3$ years, with probability of $\hat{q} = 53\%$ being a positive jump. For the stochastic AK model with standard TFP, relatively more jumps are identified as being negative (model I), $\hat{q} = 34.5\%$, with similar jump-sizes. In contrast, the other models imply a tendency towards positive jumps, $\hat{q} > 50\%$, and the jump-size is more pronounced for negative rare events.

Table 2 shows the results for industrial production of major US manufacturing industries. There are two main reasons for this approach. First, though industrial production contains intermediate production and thus contributes only minor parts to the total output, the index does play an important role in assessing the state of the economy. Second, the index is available at a monthly basis which makes it very attractive for estimating jumps. One caveat is that industrial production is more volatile than output at the quarterly basis, and could thus overemphasize the role of jumps in total output (cf. Tables 1 and 3).

For illustration, the MLE clearly suggests evidence of jumps in the data for the stochastic AK model with standard TFP (model I). In words, the point estimates suggest that jumps in aggregate IP occur on average every $1/\hat{\lambda} = 1.64$ years. With probability of $\hat{q} = 21\%$ it is a positive jump equivalent to an annualized increase of $\hat{\nu}_s = 2.38$ percentage points, whereas with probability of 79% it is a negative jump by 2.41 percentage points. Imposing the parameter restriction $\mu = (1 - e^{\upsilon_s} q - e^{-\nu_d}(1 - q))\lambda$ yields point estimates for the jump-sizes $\hat{\nu}_s = .0238$ and $\hat{\nu}_d = .0241$, the arrival rate $\hat{\lambda} = .5923$, the volatility estimate $\hat{\eta} = .0277$, the implied drift component $\hat{\mu} = .0077$, and the success probability $\hat{q} = .2254$. Statistically, we cannot distinguish between both models at conventional significance levels. In other words, the stochastic AK model with standard TFP is *not* rejected by the data.

To sum up, we find strong evidence for non-normalities in the form of jumps in aggregate US series for real GDP and industrial production. From this point of view, we establish that jumps in macroeconomics are not only caused by 'low-probability disasters' such as armed conflicts or financial crises with calibrated arrival rate $\lambda = .017$ causing a decline in economic activity of $\nu_d = .15$ as in Barro (2006, p.831). We find that jumps are a salient feature for our class of continuous-time DSGE models. Even after deflating the arrival rates of quarterly real GDP by a factor of 2.5 (which seems a conservative rule of thumb given the results for IP data), the estimated arrivals are roughly at business cycle frequency with jump-sizes only just higher than the standard deviation of the Normal innovations.

6 Conclusion

In this paper we formulate and solve continuous-time DSGE models where the economies can be non-normal and/or non-linear. For reasonable parametric restrictions we obtain explicit solutions. This feature allows us to derive the transition densities and thus the likelihood function in closed form, which then is evaluated for estimation. Hence the continuous-time formulation can make it simpler to compute and estimate the structural parameters.

In Monte Carlo experiments we show that the structural parameters of the DSGE models can be recovered for plausible values and observation frequencies of macro-series. We propose two estimation strategies. First, we use an *exact* estimation approach exploiting equilibrium conditions to filter model-specific dynamics from observed growth rates. Second, we propose an *approximate* strategy which uses equilibrium moment conditions neglecting model-specific

Table 2: US monthly IP (manufacturing sector) from 1960:01 to 2008:12

Model		*Parameter estimates*						*Value*
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$	$\ell_N(\theta)$
eq. (19)	*I*	0.0238	0.0241	0.6110	0.0277	0.0058	0.2054	1907.5
		(0.0029)	(0.0011)	(0.0815)	(0.0004)	(0.0015)	(0.0558)	(46.31)
	Ia	0.0161	0.0240	0.6844	0.0243	0.0178	0.4592	1979.8
		(0.0032)	(0.0008)	(0.1725)	(0.0004)	(0.0019)	(0.1218)	(58.16)
	Ib	0.0157	0.0238	0.7218	0.0234	0.0240	0.4891	1996.2
		(0.0024)	(0.0007)	(0.1589)	(0.0004)	(0.0018)	(0.1006)	(64.76)
eq. (20)	*II*	0.0154	0.0226	0.9477	0.0224	0.0186	0.4940	2002.3
		(0.0015)	(0.0007)	(0.1440)	(0.0004)	(0.0014)	(0.0676)	(76.94)
	IIa	0.0200	0.0250	0.6256	0.0233	0.0201	0.3758	2001.4
				(0.0633)	(0.0003)	(0.0009)	(0.0520)	(75.13)

Notes: This table reports the ML estimates for the following model specifications and calibrations: using the *exact* approach, for the standard AK (model *I*, $\alpha = 1$), the stationary AK (model *Ia* $c_1 = .5$ and *Ib* $c_1 = .7$) and the neoclassical model with a linear-policy function (model *Ia* $\alpha = .5$ and *Ib* $\alpha = .3$). Using the *approximate* approach, it reports the ML estimates for the stationary AK (model *II*, $c_1 = .5$) and the neoclassical model (model *II* $\alpha = .5$), as well as for fixed jump-size (model *IIa*). Standard errors and likelihood-ratio tests (8) are in parentheses ($\Delta = 1/12$; $N = 588$; $N = 586$ for the *exact* approach).

Table 3: US quarterly IP (manufacturing sector) from 1960:Q1 to 2008:Q4

Model		*Parameter estimates*						*Value*
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$	$\ell_N(\theta)$
eq. (19)	*I*	0.0207	0.0290	0.6953	0.0199	0.0038	0.4207	554.2
		(0.0060)	(0.0032)	(0.2995)	(0.0015)	(0.0035)	(0.1957)	(32.01)
	Ia	0.0154	0.0288	1.5202	0.0168	0.0089	0.6333	545.9
		(0.0037)	(0.0017)	(0.7012)	(0.0019)	(0.0063)	(0.1484)	(33.87)
	Ib	0.0164	0.0276	1.6823	0.0156	0.0179	0.5937	540.9
		(0.0028)	(0.0011)	(0.4611)	(0.0014)	(0.0040)	(0.0939)	(35.57)
eq. (20)	*II*	0.0201	0.0282	1.5624	0.0160	0.0217	0.5043	531.9
		(0.0017)	(0.0010)	(0.2112)	(0.0009)	(0.0024)	(0.0549)	(40.57)
	IIa	0.0250	0.0350	0.7777	0.0218	0.0255	0.3747	527.7
				(0.1125)	(0.0010)	(0.0023)	(0.0665)	(32.31)

Notes: This table reports the ML estimates for the following model specifications and calibrations: using the *exact* approach, for the standard AK (model *I*, $\alpha = 1$), the stationary AK (model *Ia* $c_1 = .5$ and *Ib* $c_1 = .7$) and the neoclassical model with a linear-policy function (model *Ia* $\alpha = .5$ and *Ib* $\alpha = .3$). Using the *approximate* approach, it reports the ML estimates for the stationary AK (model *II*, $c_1 = .5$) and the neoclassical model (model *II* $\alpha = .5$), as well as for fixed jump-size (model *IIa*). Standard errors and likelihood-ratio tests (8) are in parentheses ($\Delta = 1/4$; $N = 196$; $N = 195$ for the *exact* approach).

dynamics. Given our closed-form solutions we can address the effect of the approximation error on the parameter estimates. In our experiments, the exact method does not necessarily improve on the accuracy of estimates if the sample size is small or moderate.

In the empirical part of this paper we estimate our DSGE models using likelihood-based inference. We find strong evidence of non-normalities in the form of jumps in aggregate US data. Because of the availability of closed-form transition densities, standard likelihood-ratio tests are used to test for the presence of jumps. The null hypothesis of no jumps is rejected at conventional levels of significance for quarterly and monthly frequency of observation. Estimated jumps are found to be much more frequent and smaller than usually calibrated in models of rare disasters (as in Barro 2006).

There is a number of interesting and promising directions for future research projects. From an empirical perspective, an extension of our setup to time-varying volatility could be very useful to account for other forms of non-normalities which have been found in the literature (Fernández-Villaverde and Rubio-Ramírez 2007, Justiniano and Primiceri 2008). From an econometric perspective, estimating continuous-time DSGE in macroeconomics without closed-form solutions remains a major challenge, which could be tackled, e.g., using closed-form sequences of approximations to the transition density (Aït-Sahalia 2002).

A Appendix

A.1 The Bellman equation

Following Bellman's idea, the optimal program (15) requires (Chang 1988, Sennewald 2007)

$$\rho V(a_0, A_0) = \max_{c_0} \left\{ u(c_0) + \frac{1}{dt} E_0 dV(a_0, A_0) \right\}. \tag{36}$$

Because the stochastic processes B_t, N_t and J_t are independent, Itô's formula yields

$$\begin{aligned} dV &= V_a da_t + V_A(dA_t - (\exp(J_t) - 1) A_{t-} dN_t) + \tfrac{1}{2} V_{AA} \eta(A_t)^2 dt + (V(a_t, A_t) - V(a_t, A_{t-}))\, dN_t \\ &= V_a((r_t - \delta)a_t + w_t - c_t)dt + V_A \mu(A_t)dt + V_A \eta(A_t) dB_t + \tfrac{1}{2} V_{AA} \eta(A_t)^2 dt \\ &\quad + (V(a_t, \exp(J_t)A_{t-}) - V(a_t, A_{t-}))\, dN_t, \end{aligned}$$

where we used $A_t = \exp(J_t)A_{t-}$, that is the level immediately after a jump. By construction $a_t = a_{t-}$ as for any continuous-state process. Apply now the operator $E_0(\cdot)$ to the integral equation, and use the property of the Itô stochastic integral, $E_0 \int_0^t f(s, B_s) dB_s = 0$, and the martingale property of $\int_0^t f(s, N_s)\, dN_s - \lambda \int_0^t f(s, N_s)\, ds$. We insert the resulting expression for $E_0 dV(a_0, A_0)$ into the Bellman equation (36) and obtain

$$\rho V(a_t, A_t) = \max_{c_t} \big\{ u(c_t) + V_a((r_t - \delta)a_t + w_t - c_t) + V_A \mu(A_t) \\ + \tfrac{1}{2} V_{AA} \eta(A_t)^2 + \left(qV(a_t, e^{\nu_s} A_t) + (1-q)V(a_t, e^{-\nu_d} A_t) - V(a_t, A_t) \right) \lambda \big\} \quad (37)$$

for any $t \in [0, \infty)$, where we inserted $E_J(\cdot)$ as the expectation operator with respect to J_t,

$$E_J \left\{ V(a_t, \exp(J_t) A_{t-}) \right\} \equiv qV(a_t, e^{\nu_s} A_{t-}) + (1-q)V(a_t, e^{-\nu_d} A_{t-}). \quad (38)$$

Because (37) is a necessary condition for optimality, we obtain the first-order condition.

A.2 Proof of Proposition 3.1

The idea of this proof is to show that with a candidate solution, both the maximized Bellman equation (17), and the first-order condition (16) are fulfilled (Chang 1988, Sennewald 2007). We start with an educated guess of the value function, and then derive conditions under which it actually is the unique solution to the control problem

$$V(a_t, A_t) = \mathbb{C}_1 \ln a_t + f(A_t). \quad (39)$$

From the first-order condition (16), for the case of logarithmic utility ($\sigma = 1$), optimal consumption is a constant fraction of wealth,

$$c_t^{-1} = \mathbb{C}_1 a_t^{-1} \quad \Leftrightarrow \quad c_t = \mathbb{C}_1^{-1} a_t.$$

Now use the maximized Bellman equation (17), and insert the solution candidate

$$\begin{aligned} \rho V(a_t, A_t) &= \ln c(a_t) + V_a((r_t - \delta)a_t - c(a_t)) + V_A \mu A_t + \tfrac{1}{2} V_{AA} A_t^2 \eta^2 + E_J \left\{ f(A_t) - f(A_{t-}) \right\} \lambda \\ \Leftrightarrow \ \rho \mathbb{C}_1 \ln a_t &= \ln a_t - \ln \mathbb{C}_1 + \mathbb{C}_1 a_t^{-1}((A_t - \delta)a_t - \mathbb{C}_1^{-1} a_t) + f_A \mu A_t + \tfrac{1}{2} f_{AA} A_t^2 \eta^2 - \rho f(A_t) \\ &\quad + E_J \left\{ f(A_t) - f(A_{t-}) \right\} \lambda. \end{aligned}$$

Choose $\rho f(A_t) = \mathbb{C}_1 A_t + f_A \mu A_t + \frac{1}{2} f_{AA} A_t^2 \eta^2 + E_J\{f(A_t) - f(A_{t-})\}\lambda - \ln \mathbb{C}_1 - \mathbb{C}_1 \delta - 1$, and the equation simplifies to $\rho \mathbb{C}_1 \ln a_t = \ln a_t$. This expression indeed is a solution for $\mathbb{C}_1 = 1/\rho$. Because the economy has L representative households, $C_t = L c_t = \mathbb{C}_1^{-1} L a_t = \rho K_t$.

A.3 Proof of Proposition 3.4

The idea of this proof follows Section A.2. An educated guess of the value function is

$$V(a_t, A_t) = \frac{\mathbb{C}_1 a_t^{1-\sigma}}{1-\sigma} + f(A_t). \tag{40}$$

From (16), optimal consumption per effective worker is a constant fraction of wealth,

$$c_t^{-\sigma} = \mathbb{C}_1 a_t^{-\sigma} \quad \Leftrightarrow \quad c_t = \mathbb{C}_1^{-1/\sigma} a_t.$$

Now use the maximized Bellman equation (17), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L^{-\alpha}$, together with the transformation $K_t \equiv L a_t$, and insert the solution candidate,

$$\begin{aligned}
\rho V(a_t, A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{1-\sigma}}{1-\sigma} + V_a((r_t - \delta) a_t + w_t - c(a_t)) + V_A \mu(A_t) + \tfrac{1}{2} V_{AA} \eta(A_t)^2 \\
&\quad + E_J \left(V(a_t, A_t) - V(a_t, A_{t-}) \right) \lambda \\
\Leftrightarrow \ \rho \frac{\mathbb{C}_1 a_t^{1-\sigma}}{1-\sigma} + g(A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{1-\sigma}}{1-\sigma} + \mathbb{C}_1 a_t^{-\sigma}((\alpha A_t a_t^{\alpha-1} - \delta) a_t + (1-\alpha) A_t a_t^{\alpha} - \mathbb{C}_1^{-1/\sigma} a_t) \\
&= \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{1-\sigma}}{1-\sigma} + \mathbb{C}_1 a_t^{-\sigma}(A_t a_t^{\alpha} - \delta a_t - \mathbb{C}_1^{-1/\sigma} a_t),
\end{aligned}$$

where $g(A_t) \equiv \rho f(A_t) - f_A \mu(A_t) - \frac{1}{2} f_{AA} \eta(A_t)^2 - E_J \{ f(A_t) - f(A_{t-}) \} \lambda$. When imposing the condition $\alpha = \sigma$ and $g(A_t) = \mathbb{C}_1 A_t$ the equation can be simplified to

$$\begin{aligned}
\rho \frac{\mathbb{C}_1 a_t^{1-\sigma}}{1-\sigma} + g(A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{1-\sigma}}{1-\sigma} + \mathbb{C}_1 (A_t - \delta a_t^{1-\sigma} - \mathbb{C}_1^{-1/\sigma} a_t^{1-\sigma}) \\
\Leftrightarrow \ \rho a_t^{1-\sigma} &= \sigma \mathbb{C}_1^{-\frac{1}{\sigma}} a_t^{1-\sigma} - (1-\sigma) \delta a_t^{1-\sigma} \quad \Rightarrow \quad \mathbb{C}_1^{-1/\sigma} = \frac{\rho + (1-\sigma)\delta}{\sigma},
\end{aligned}$$

which finally proves that $\phi = \mathbb{C}_1^{-1/\sigma}$.

A.4 Proof of Proposition 3.5

The idea of this proof follows Section A.2. An educated guess of the value function is

$$V(a_t, A_t) = \frac{\mathbb{C}_1 a_t^{1-\alpha\sigma}}{1-\alpha\sigma} A_t^{-\sigma}. \tag{41}$$

From (16), optimal consumption per effective worker is a constant fraction of income,

$$c_t^{-\sigma} = \mathbb{C}_1 a_t^{-\alpha\sigma} A_t^{-\sigma} \quad \Leftrightarrow \quad c_t = \mathbb{C}_1^{-1/\sigma} a_t^{\alpha} A_t.$$

Now use the maximized Bellman equation (17), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L^{-\alpha}$, together with the transformation $K_t \equiv L a_t$, and insert the solution candidate,

$$
\begin{aligned}
\rho V(a_t, A_t) \;\; = \;\; & \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{\alpha-\alpha\sigma} A_t^{1-\sigma}}{1-\sigma} + V_a((r_t - \delta) a_t + w_t - c(a_t, A_t)) + V_A \mu(A_t) + \tfrac{1}{2} V_{AA} \eta(A_t)^2 \\
& + \left(E_J \left\{\exp(-\sigma J_t)\right\} - 1\right) \mathbb{C}_1 a_t^{1-\alpha\sigma} / (1-\alpha\sigma) A_t^{-\sigma} \lambda
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\rho \frac{\mathbb{C}_1 a_t^{1-\alpha\sigma}}{1-\alpha\sigma} A_t^{-\sigma} \;\; = \;\; & \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{\alpha-\alpha\sigma} A_t^{1-\sigma}}{1-\sigma} + \mathbb{C}_1 a_t^{-\alpha\sigma} A_t^{-\sigma} (\alpha A_t a_t^{\alpha} - \delta a_t + (1-\alpha) A_t a_t^{\alpha} - \mathbb{C}_1^{-1/\sigma} a_t^{\alpha} A_t) \\
& - \sigma \frac{\mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma-1}}{1-\alpha\sigma} \mu(A_t) + \frac{\frac{1}{2}\sigma(1+\sigma) \mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma-2}}{1-\alpha\sigma} \eta(A_t)^2 \\
& + \left(E_J \left\{\exp(-\sigma J_t)\right\} - 1\right) \mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma} \lambda / (1-\alpha\sigma).
\end{aligned}
$$

Collecting terms gives

$$
\begin{aligned}
& \rho \frac{\mathbb{C}_1 a_t^{1-\alpha\sigma}}{1-\alpha\sigma} A_t^{-\sigma} + \sigma \frac{\mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma-1}}{1-\alpha\sigma} \mu(A_t) - \frac{\frac{1}{2}\sigma(1+\sigma) \mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma-2}}{1-\alpha\sigma} \eta(A_t)^2 = \\
& \quad \frac{\mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{\alpha-\alpha\sigma} A_t^{1-\sigma}}{1-\sigma} + \mathbb{C}_1 a_t^{\alpha-\alpha\sigma} A_t^{1-\sigma} - \delta \mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma} - \mathbb{C}_1^{-\frac{1-\sigma}{\sigma}} a_t^{\alpha-\alpha\sigma} A_t^{1-\sigma} \\
& \quad + \left(E_J \left\{\exp(-\sigma J_t)\right\} - 1\right) \mathbb{C}_1 a_t^{1-\alpha\sigma} A_t^{-\sigma} \lambda / (1-\alpha\sigma) \\
\Leftrightarrow \;\; & \rho + \sigma \mu(A_t)/A_t - \tfrac{1}{2}\sigma(1+\sigma) \eta(A_t)^2 / A_t^2 + (1-\alpha\sigma)\delta - \left(E_J \left\{\exp(-\sigma J_t)\right\} - 1\right) \lambda = \\
& \quad \left(1 - \sigma + \sigma \mathbb{C}_1^{-1/\sigma}\right) \frac{1-\alpha\sigma}{1-\sigma} a_t^{\alpha-1} A_t
\end{aligned}
$$

which has a solution for $\mathbb{C}_1^{-1/\sigma} = (\sigma - 1)/\sigma$ and

$$
\rho = -\sigma \mu(A_t)/A_t + \tfrac{1}{2}\sigma(1+\sigma) \eta(A_t)^2 / A_t^2 - (1-\alpha\sigma)\delta + \left(E_J \left\{\exp(-\sigma J_t)\right\} - 1\right) \lambda. \qquad (42)
$$

The condition implicitly restricts $\mu(A_t)$ and $\sigma(A_t)$ to the set of functions

$$
\mu(A_t)/A_t - \tfrac{1}{2}(1+\sigma) \eta(A_t)^2 / A_t^2 \;\equiv\; \bar{\mu} \in \mathbb{R}.
$$

For reasonable parametric calibrations equation (42) is satisfied. Though being a special case, a Keynesian consumption function could be an admissible policy function for the neoclassical model (cf. also Chang 1988). Its plausibility is an empirical question.

A.5 Boundedness of life-time utility in the neoclassical model

For our solutions the integral (12) has to exist. This section illustrates how to obtain the boundedness condition for Proposition 3.4. Other solutions follow a similar approach. Given the parameter restriction $\alpha = \sigma$, equation (11) simplifies to $dK_t = (A_t K_t^\alpha L^{1-\alpha} - (\delta+\phi)K_t)dt$ which now can be solved explicitly. Using the transformation $c_t^{1-\sigma} = (\phi K_t/L)^{1-\alpha}$,

$$\begin{aligned} dc_t^{1-\sigma} &= (1-\alpha)K_t^{-\alpha}\phi^{1-\alpha}(A_t K_t^\alpha - (\delta+\phi)K_t/L^{1-\alpha})dt \\ &= (1-\alpha)(A_t\phi^{1-\alpha} - (\delta+\phi)c_t^{1-\sigma})dt, \end{aligned}$$

which has the solution $c_t^{1-\sigma} = e^{-(1-\alpha)(\delta+\phi)t}(c_0^{1-\sigma} + (1-\alpha)\phi^{1-\alpha}\int_0^t e^{(1-\alpha)(\delta+\phi)s}A_s ds)$. Passing the solution to the integral (12) we may write life-time utility as

$$U_0 = \int_0^\infty e^{-\rho t-(1-\alpha)(\delta+\phi)t}\left(c_0^{1-\sigma}/(1-\sigma) + \phi^{1-\alpha}\int_0^t e^{(1-\alpha)(\delta+\phi)s}E_0(A_s)ds\right)dt.$$

Now insert $E_0(A_t) = A_0 e^{(\mu+\lambda(e^{\upsilon_s}q+e^{-\nu_d}(1-q)-1))t}$, and collect terms to obtain

$$U_0 = \int_0^\infty e^{-\rho t-(1-\alpha)(\delta+\phi)t}\left(\frac{c_0^{1-\sigma}}{1-\sigma} + \frac{\phi^{1-\alpha}A_0\left(e^{(1-\alpha)(\delta+\phi)t+\left(\mu+\lambda(e^{\upsilon_s}q+e^{-\nu_d}(1-q)-1)\right)t}-1\right)}{(1-\alpha)(\delta+\phi)+\mu+\lambda(e^{\upsilon_s}q+e^{-\nu_d}(1-q)-1)}\right)dt,$$

which gives the boundedness condition $\rho > \mu + \lambda(e^{\upsilon_s}q + e^{-\nu_d}(1-q)-1)$. By definition, this integral is $V(a_0, A_0)$ in (40), which implies $f(A_0)$ where f_A is a constant, and $f_{AA} = 0$.

A.6 Moments of capital rewards in the neoclassical model

In a seminal paper Merton (1975) shows that the output-to-capital ratio in the Solow model under Normal uncertainty has a Gamma distribution. Consider the output-to-capital ratio allowing for Poisson uncertainty. Suppose that r_t has a limiting distribution, such that the sequence $\{r_t\}_{t=t_0}^\infty$ converges in distribution to a random variable r,

$$r_t \xrightarrow{\mathcal{D}} r \quad \text{where} \quad 0 < r_t < \infty. \tag{43}$$

In the paper we show that (31) for the standard TFP process (30) can be written as

$$dr_t = c_1 r_t (c_2 - r_t)\,dt + \eta r_t dB_t + (\exp(J_t)-1)r_{t-}dN_t. \tag{44}$$

Because the SDE is a reducible geometric-reverting jump-diffusion process, it can be solved explicitly. Due to the non-linearity, obtaining the moments directly from the solution does not seem promising. We use the smooth transformation $\ln r_t$,

$$\ln r_t \xrightarrow{\mathcal{D}} \ln r \quad \text{where} \quad -\infty < \ln r_t < \infty \tag{45}$$

to obtain $d\ln r_t = c_1(c_2 - r_t)dt - \frac{1}{2}\eta^2 dt + \eta dB_t + J_t dN_t$ which has the solution

$$\ln r_t - \ln r_{t_0} = \int_{t_0}^{t} \left(c_1(c_2 - r_s) - \tfrac{1}{2}\eta^2\right) ds + \eta(B_t - B_{t_0}) + \int_{t_0}^{t} J_t dN_t.$$

Employing the property that $\ln r_t$ and $\ln r_{t-\Delta}$ share the same asymptotic mean as from (45),

$$\begin{aligned} 0 &= \left(c_1 c_2 - \tfrac{1}{2}\eta^2\right)\Delta - c_1 \lim_{t\to\infty}\int_{t-\Delta}^{t} E_0(r_s)ds + \nu \lim_{t\to\infty} E_0(N_t - N_{t-\Delta}) \\ &= \left(c_1 c_2 - \tfrac{1}{2}\eta^2 + \nu\lambda\right)\Delta - c_1 \lim_{t\to\infty}\int_{t-\Delta}^{t} E_0(r_s)ds \\ \Rightarrow \quad E(r) &\equiv \lim_{t\to\infty} E_0(r_t) = \frac{c_1 c_2 - \frac{1}{2}\eta^2 + \nu\lambda}{c_1}. \end{aligned} \tag{46}$$

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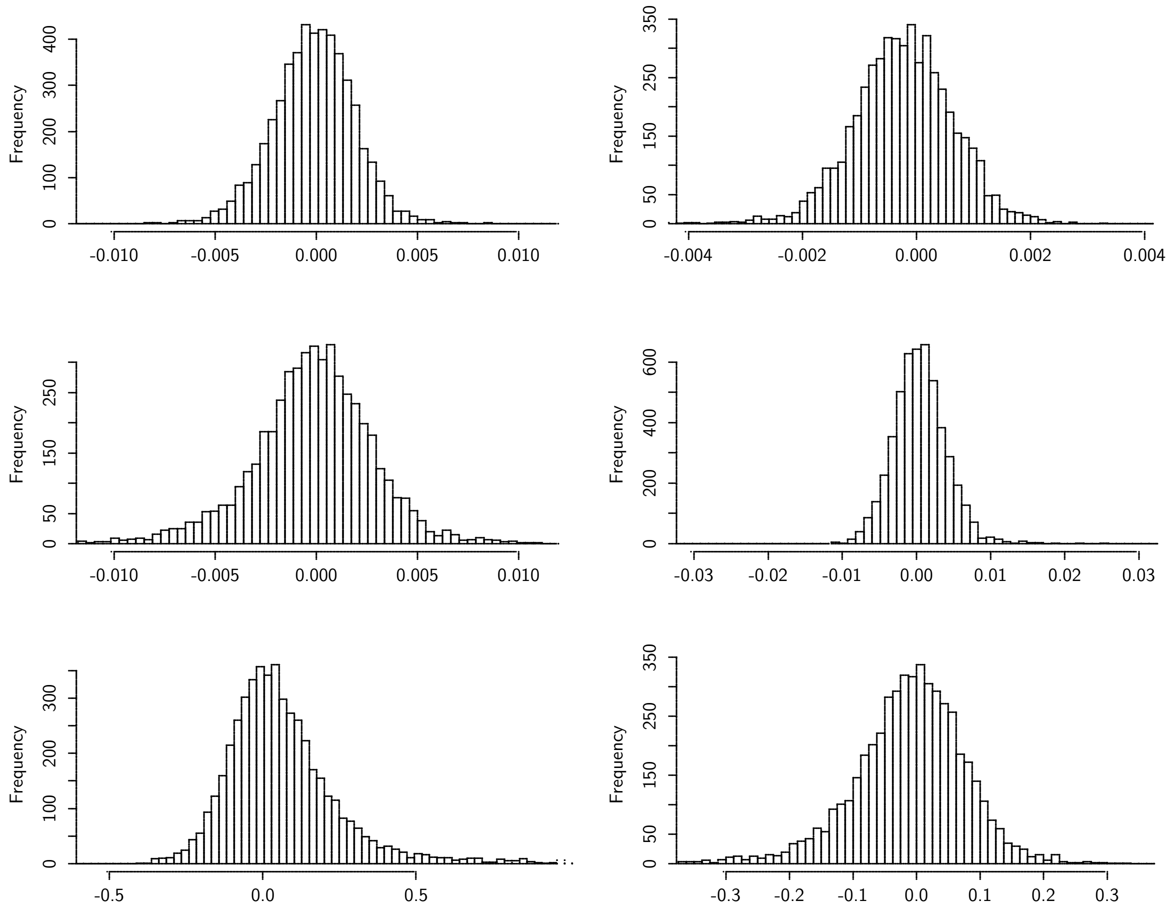
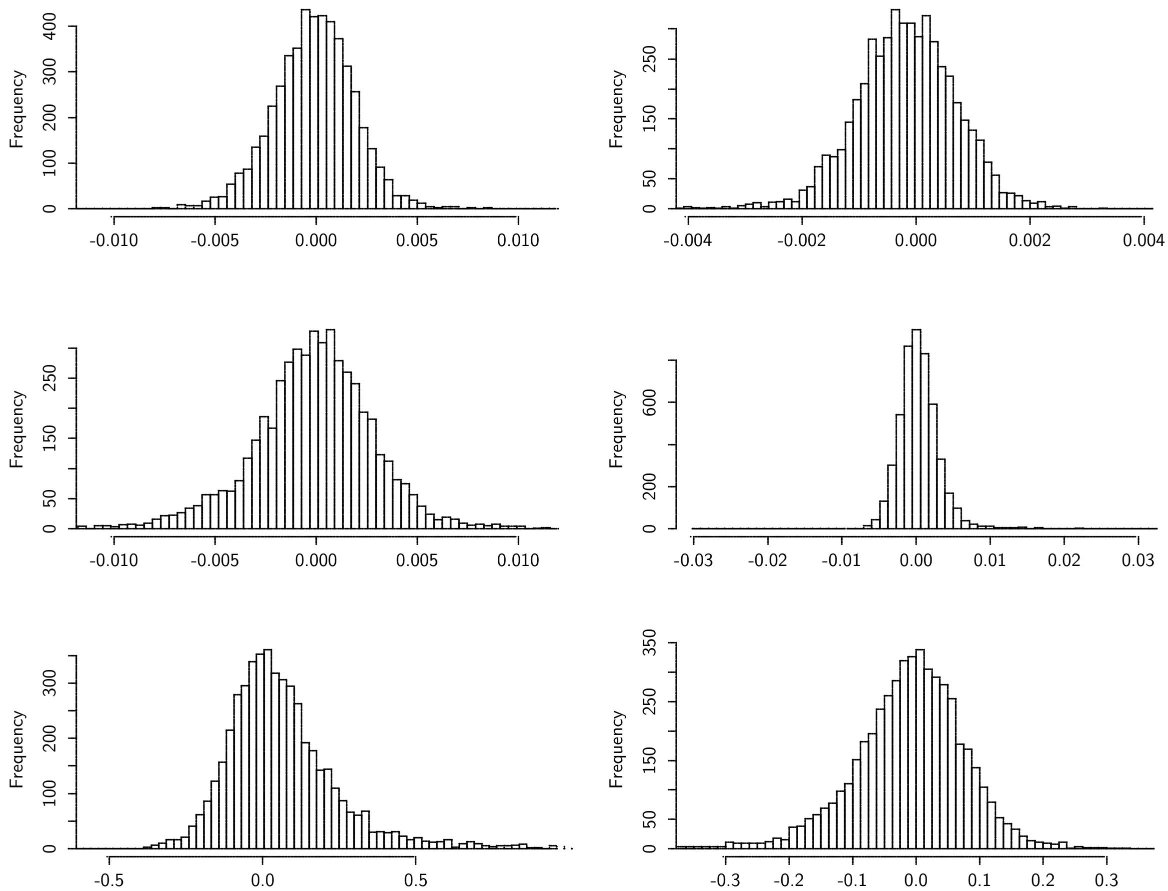
Figure 1: Finite sample distribution of estimation errors ($\Delta = 1/10$, model I)



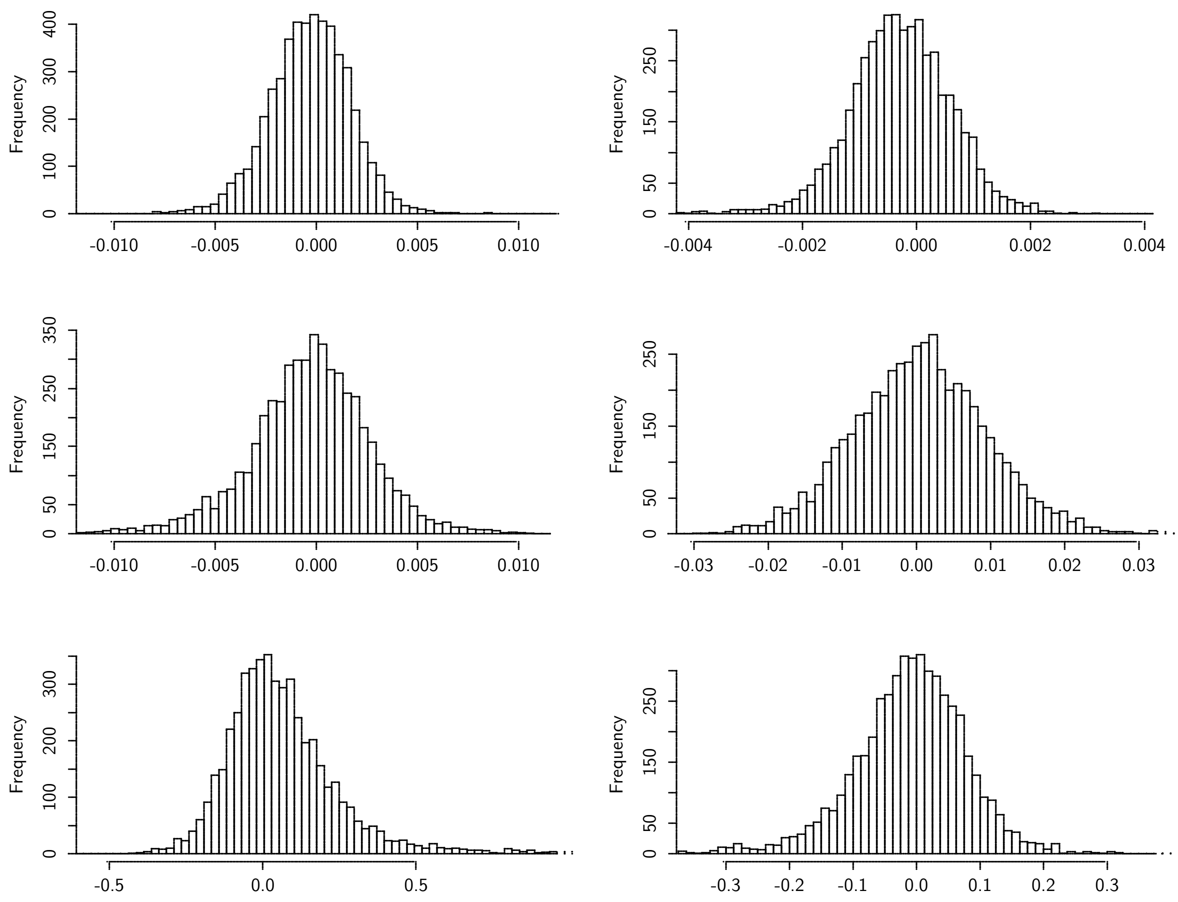
Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations for ν_s, ν_d, λ, η, μ, and q (column by column, from top left to bottom right), for the *exact* approach (19) (sample parameters and ML estimates are summarized in Table 4, model I).

Figure 2: Finite sample distribution of estimation errors ($\Delta = 1/10$, model II)



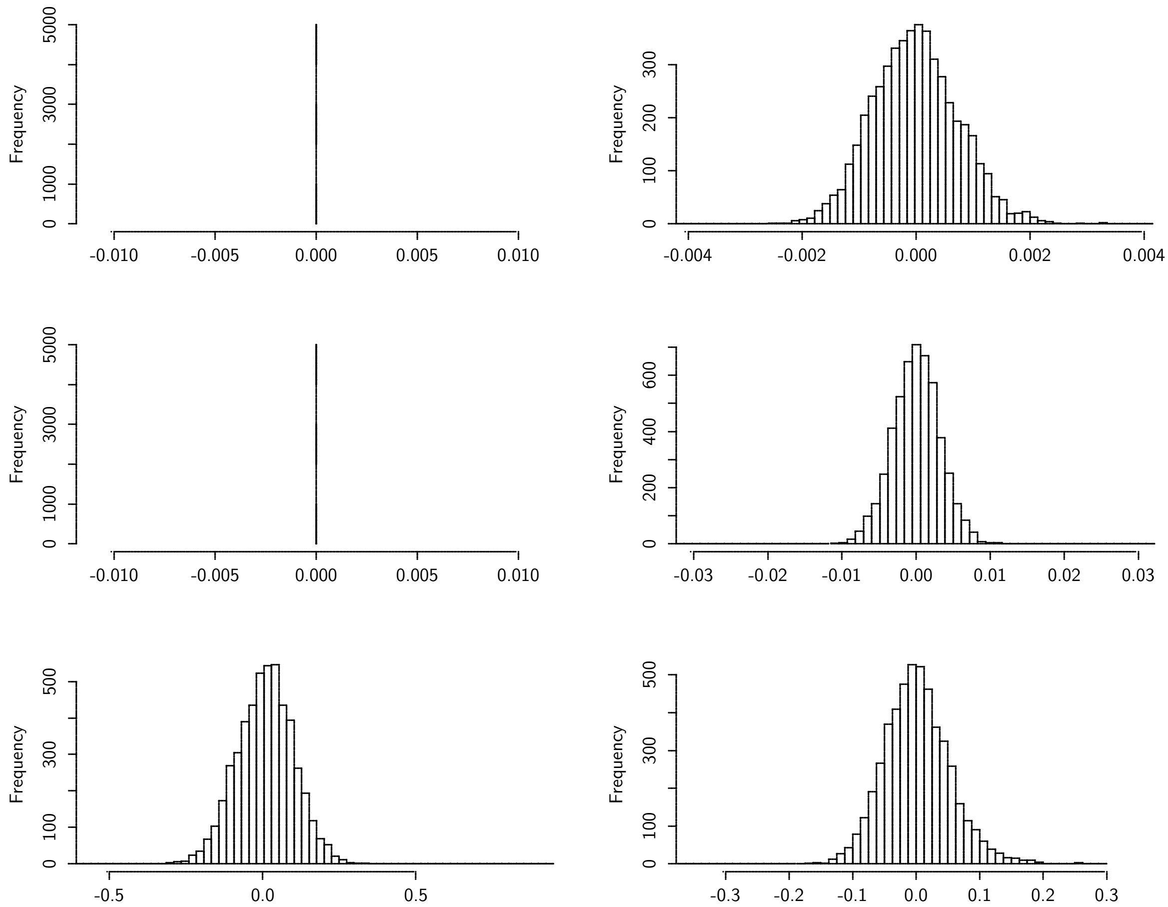
Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations for ν_s, ν_d, λ, η, μ, and q (column by column, from top left to bottom right), for the *approximative* approach (20) (sample parameters and ML estimates are summarized in Table 4, model II).

Figure 3: Finite sample distribution of estimation errors ($\Delta = 1/10$, model *IIa*)



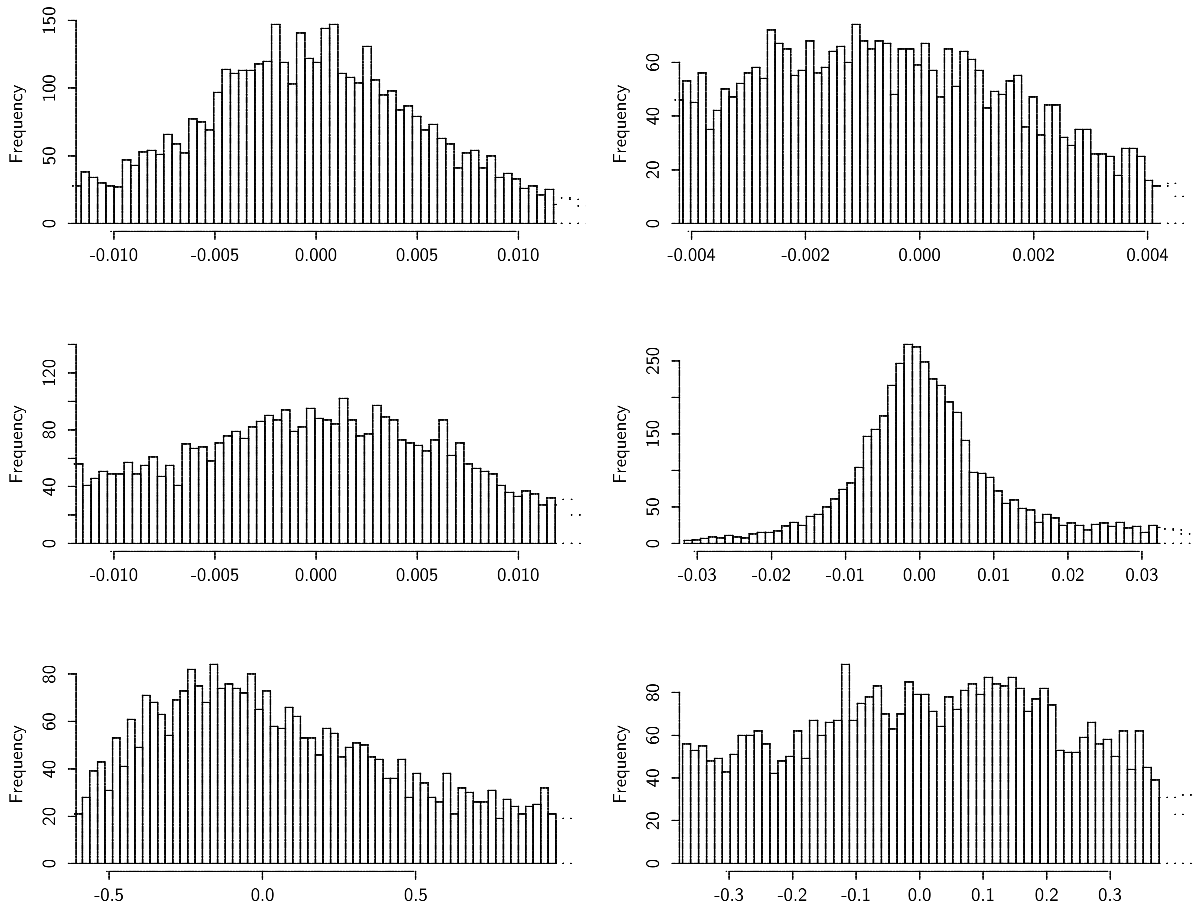
Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations for ν_s, ν_d, λ, η, μ, and q (column by column, from top left to bottom right), for the *approximative* approach (20) (sample parameters and ML estimates are summarized in Table 4, model *IIa*).

Figure 4: Finite sample distribution of estimation errors ($\Delta = 1/10$, model Ia)



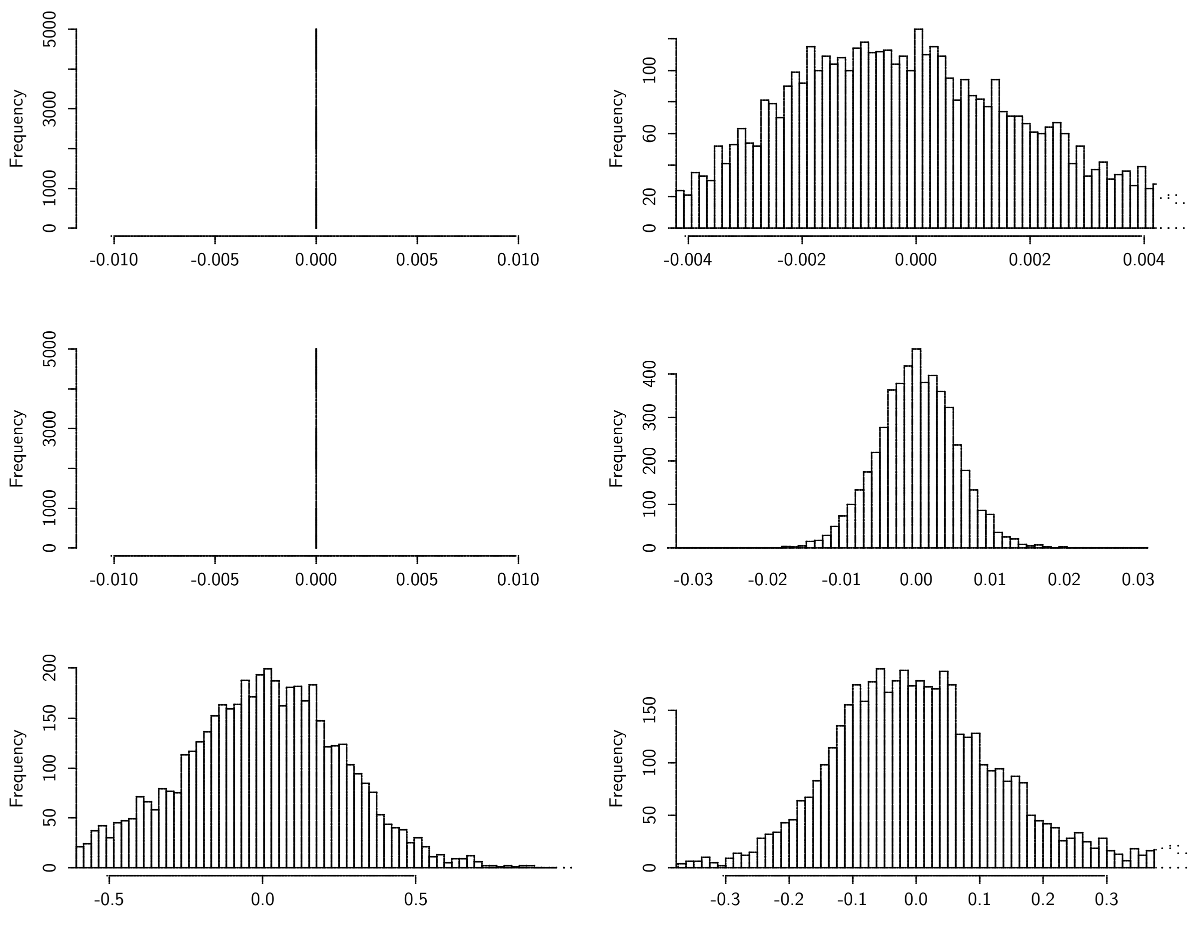
Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations (fixed jump-size), $\bar{\nu}_s$, $\bar{\nu}_d$, λ, η, μ, and q (column by column, from top left to bottom right), for the *exact* approach (19) (sample parameters and ML estimates are summarized in Table 4, model Ia).

Figure 5: Finite sample distribution of estimation errors ($\Delta = 1/4$, model I)



Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations for ν_s, ν_d, λ, η, μ, and q (column by column, from top left to bottom right), for the *exact* approach (19) (sample parameters and ML estimates are summarized in Table 6, model I).

Figure 6: Finite sample distribution of estimation errors ($\Delta = 1/4$, model Ia)



Notes: These figures report histograms of differences of ML estimates and sample parameters of $M = 5000$ Monte Carlo simulations (fixed jump-size), $\bar{\nu}_s$, $\bar{\nu}_d$, λ, η, μ, and q (column by column, from top left to bottom right), for the *exact* approach (19) (sample parameters and ML estimates are summarized in Table 6, model Ia).

Table 4: Comparison of Monte Carlo estimates and sample averages ($\Delta = 1/10$)

Model		*Parameter estimates averages*					
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$
Sample		0.0250	0.0200	0.7905	0.0200	0.0100	0.5011
		(0.0000)	(0.0000)	(0.1193)	(0.0002)	(0.0027)	(0.0749)
eq. (19)	*I*	0.0249	0.0198	0.8971	0.0198	0.0106	0.4887
		(0.0020)	(0.0031)	(0.5856)	(0.0010)	(0.0063)	(0.1151)
	Ia	0.0250	0.0200	0.7993	0.0200	0.0100	0.5028
				(0.1455)	(0.0008)	(0.0056)	(0.0908)
eq. (20)	*II*	0.0249	0.0198	0.8929	0.0198	0.0105	0.4890
		(0.0020)	(0.0031)	(0.4937)	(0.0010)	(0.0049)	(0.1151)
	IIa	0.0247	0.0197	0.9017	0.0197	0.0105	0.4890
		(0.0020)	(0.0030)	(0.6788)	(0.0010)	(0.0118)	(0.1158)
	IIb	0.0250	0.0200	0.7896	0.0200	0.0099	0.5045
				(0.1451)	(0.0008)	(0.0114)	(0.0916)

Notes: This table reports the averages of $M = 5000$ Monte Carlo simulations. It compares the mean of ML estimates to sample averages for the following model specifications: using the *exact* approach (model *I*) and for fixed jump-size (model *Ia*); using the *approximate* approach for different levels of the speed of reversion (model *II*, $c_1 = .5$ and *IIa*, $c_1 = 3$) and for fixed jump-size (model *IIb*, $c_1 = 3$). Sampling distribution standard errors are in parentheses ($\Delta = 1/10$; $N = 580$).

Table 5: Comparison of Monte Carlo estimates and sample averages (.95 quantile, $\Delta = 1/10$)

Model		*Parameter estimates averages*					
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$
Sample		0.0250	0.0200	0.7905	0.0200	0.0100	0.5011
		(0.0000)	(0.0000)	(0.1193)	(0.0002)	(0.0027)	(0.0749)
eq. (19)	*I*	0.0249	0.0201	0.8318	0.0198	0.0102	0.4987
		(0.0020)	(0.0027)	(0.1784)	(0.0009)	(0.0057)	(0.1057)
eq. (20)	*II*	0.0250	0.0202	0.8320	0.0199	0.0101	0.4987
		(0.0020)	(0.0027)	(0.1787)	(0.0009)	(0.0042)	(0.1063)
	IIa	0.0248	0.0200	0.8312	0.0198	0.0102	0.4989
		(0.0020)	(0.0026)	(0.1800)	(0.0009)	(0.0114)	(0.1064)

Notes: This table reports the averages of the .95 quantile of $M = 5000$ Monte Carlo simulations ordered by $\hat{\lambda}$. It compares averages of ML estimates for the following model specifications: using the *exact* approach (model *I*); using the *approximate* approach for different levels of the speed of reversion (model *II*, $c_1 = .5$ and *IIa*, $c_1 = 3$). Sampling distribution standard errors are in parentheses ($\Delta = 1/10$; $N = 580$).

Table 6: Comparison of Monte Carlo estimates and sample averages ($\Delta = 1/4$)

Model		*Parameter estimates averages*					
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$
Sample		0.0250	0.0200	0.7905	0.0200	0.0100	0.5011
		(0.0000)	(0.0000)	(0.1193)	(0.0002)	(0.0027)	(0.0749)
eq. (19)	*I*	0.0248	0.0197	2.5648	0.0168	0.0148	0.4733
		(0.0082)	(0.0101)	(3.0406)	(0.0050)	(0.0219)	(0.2810)
	Ia	0.0250	0.0200	0.7744	0.0201	0.0101	0.5258
				(0.2981)	(0.0025)	(0.0070)	(0.1980)
eq. (20)	*II*	0.0250	0.0196	2.5817	0.0169	0.0147	0.4690
		(0.0084)	(0.0102)	(3.0280)	(0.0051)	(0.0216)	(0.2758)
	IIa	0.0244	0.0193	2.6364	0.0167	0.0146	0.4735
		(0.0083)	(0.0103)	(3.0831)	(0.0050)	(0.0245)	(0.2857)
	IIb	0.0250	0.0200	0.7338	0.0201	0.0100	0.5325
				(0.2927)	(0.0025)	(0.0121)	(0.2039)

Notes: This table reports the averages of $M = 5000$ Monte Carlo simulations. It compares the mean of ML estimates to sample averages for the following model specifications: using the *exact* approach (model *I*) and for fixed jump-size (model *Ia*); using the *approximate* approach for different levels of the speed of reversion (model *II*, $c_1 = .5$ and *IIa*, $c_1 = 3$) and for fixed jump-size (model *IIb*, $c_1 = 3$). Sampling distribution standard errors are in parentheses ($\Delta = 1/4$; $N = 232$).

Table 7: Comparison of Monte Carlo estimates and sample averages (.95 quantile, $\Delta = 1/4$)

Model		*Parameter estimates averages*					
		$\hat{\nu}_s$	$\hat{\nu}_d$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{q}$
Sample		0.0250	0.0200	0.7905	0.0200	0.0100	0.5011
		(0.0000)	(0.0000)	(0.1193)	(0.0002)	(0.0027)	(0.0749)
eq. (19)	*I*	0.0249	0.0202	2.1050	0.0172	0.0133	0.4856
		(0.0081)	(0.0099)	(2.3332)	(0.0047)	(0.0177)	(0.2687)
eq. (20)	*II*	0.0251	0.0201	2.1231	0.0173	0.0133	0.4813
		(0.0083)	(0.0099)	(2.3199)	(0.0048)	(0.0171)	(0.2631)
	IIa	0.0245	0.0198	2.1790	0.0170	0.0131	0.4861
		(0.0082)	(0.0101)	(2.4013)	(0.0048)	(0.0208)	(0.2737)

Notes: This table reports the averages of the .95 quantile of $M = 5000$ Monte Carlo simulations ordered by $\hat{\lambda}$. It compares averages of ML estimates for the following model specifications: using the *exact* approach (model *I*); using the *approximate* approach for different levels of the speed of reversion (model *II*, $c_1 = .5$ and *IIa*, $c_1 = 3$). Sampling distribution standard errors are in parentheses ($\Delta = 1/4$; $N = 232$).