Risk premia in general equilibrium

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Abstract

This paper shows that non-linearities from a neoclassical production function alone can generate time-varying, asymmetric risk premia and predictability over the business cycle. These empirical key features become relevant when we allow for non-normalities in the form of rare disasters. We employ analytical solutions of dynamic stochastic general equilibrium models, including a novel solution with endogenous labor supply, to obtain closed-form expressions for the risk premium in production economies. In contrast to an endowment economy with constant investment opportunities, the curvature of the consumption function affects the risk premium in production economies through controlling the individual's effective risk aversion.

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1 Introduction

"... the challenge now is to understand the economic forces that determine the stochastic discount factor, or put another way, the rewards that investors demand for bearing particular risks." (Campbell, 2000, p.1516)

In general equilibrium models, the 'stochastic discount factor', i.e., the stochastic process used to discount returns of any security, is not only determined by the consumption-based first-order condition, but also linked to business cycle characteristics. In macroeconomics, dynamic stochastic general equilibrium (DSGE) models have been successful in explaining co-movements in aggregate data, but relatively less progress has been made to reconcile their asset market implications with financial data (cf. Grinols and Turnovsky, 1993; Jermann, 1998, 2010; Tallarini, 2000; Lettau and Uhlig, 2000; Boldrin, Christiano and Fisher, 2001; Lettau, 2003; Campanale, Castro and Clementi, 2010).[1] One main advantage of using general equilibrium models to explain asset market phenomena is that the asset-pricing kernel is consistent with the macro dynamics.

However, surprisingly little is known about the risk premium in non-linear DSGE models, i.e., the minimum difference an individual requires to accept an uncertain rate of return, between its expected value and the certainty equivalent rate of return on saving he or she is indifferent to.[2] At least two primary questions present themselves. Which economic forces determine the risk premium in general equilibrium? What are the implications of using production based models compared to the endowment economy? This paper fills the gap by studying asset pricing implications of the prototype production economy analytically. Why is this important? We argue that a clear understanding of the risk premium can best be achieved by working out analytical solutions. These solutions are shown to be important knife-edge cases which can therefore be used to shed light on our numerical results.

In a nutshell, this paper shows that a neoclassical production function alone generates key features of the risk premium. The economic intuition is that the individual's risk aversion, excluding singular cases, is not constant in a neoclassical production economy.

We use analytical solutions of DSGE models. For this purpose we readopt formulating models in continuous time (as in Merton, 1975; Eaton, 1981; Cox, Ingersoll and Ross, 1985), which gives closed-form solutions for a broad class of models and parameter sets.[3] Recent

[1]There is an increasing interest in macro-finance DSGE models (cf. Kaltenbrunner and Lochstoer, 2006; Gourio, 2010), A survey on the intersection of macro and finance is provided in Cochrane (2008, chap. 7) .

[2]Grinols and Turnovsky (1993) and Turnovsky and Bianconi (2005) study asset pricing implications of aggregate risk and/or idiosyncratic shocks in stochastic endogenous growth models with a quasi-linear production technology. Our formulation focuses on non-linear DSGE models with transitional dynamics.

[3]Analytical solutions to DSGE models are used in Turnovsky (1993, 2000), Corsetti (1997), Wälde (2005), Turnovsky and Smith (2006), and Posch (2009) (a detailed discussion can be found in Wälde, 2010).

research emphasizes the importance of non-normalities and non-linearities in explaining business cycle dynamics for the US economy (Fernández-Villaverde and Rubio-Ramírez, 2007; Justiniano and Primiceri, 2008; Posch, 2009). Therefore, our starting point is Lucas' fruit-tree endowment economy with non-normalities in the form of rare disasters. We obtain closed-form expressions for the risk premium from the Euler equation and relate it to the market premium over a riskless rate of return. Subsequently the framework is extended to a non-linear production economy with endogenous consumption choice and labor supply. Our approach still gives closed-form expressions under parametric restrictions.

The major findings can be summarized as follows. While the endowment economy implies a constant risk premium, non-linearities in production economies can generate time-varying, asymmetric risk premia and predictability over the business cycle.[4] Although these empirical key features of the risk premium are negligible in the standard real business cycle (RBC) model, we show that they become relevant when we allow for non-normalities in the form of rare disasters (Rietz, 1988; Barro, 2006, 2009). Our results are based on the finding that the 'effective risk aversion' is not constant for non-homogeneous consumption functions, as it refers to the risk aversion of the value function (cf. Carroll and Kimball, 1996).[5] Even for constant relative risk aversion (CRRA) of the direct utility function, the individual's effective risk aversion is not necessarily constant since it refers to gambles with respect to wealth. As we show in Section 3.2, non-homogeneous consumption functions are typically found in production economies.[6]

One caveat of many discrete-time models is the difficulty to obtain analytical solutions. To some extent, it is due to the difficulty of solving these models that endowment economies, in contrast to the typically non-linear production economies used in macroeconomics, are popular for asset pricing in finance. In particular by focusing on the effects of uncertainty, the traditional approach of linearization about the non-stochastic steady state does not provide an adequate framework. Alternatively, the literature suggests either risk-sensitive objectives (Hansen, Sargent and Tallarini, 1999; Tallarini, 2000) or log-linearization methods (Campbell, 1994; Lettau, 2003). Similarly, numerical strategies employ perturbation and higher-order approximation schemes (cf. Taylor and Uhlig, 1990; Schmitt-Grohé and Uribe, 2004; Fernández-Villaverde and Rubio-Ramírez, 2006) . Although these numerical methods

[4]Both the time-varying feature and evidence that the risk premium increases more in 'bad times' than it decreases in 'good times' are found empirically (Lettau and Ludvigson, 2001; Mehra and Prescott, 2008).

[5]Any function $f(x)$ that does not have the property that for any scalar $b > 0$ there is a scalar k such that $b^k f(x) = f(bx)$ is said to be non-homogeneous. A consumption function gives optimal consumption (the control variable) as a function of the state variables.

[6]Other contributions to Mehra and Presott's (1985) equity premium puzzle for endowment economies, e.g., Epstein and Zin (1989); Abel (1990, 1999); Constantinides (1990); Campbell and Cochrane (1999); Veronesi (2004); Bansal and Yaron (2004), generate time-varying risk aversion through different channels.

usually are highly accurate locally, the effects of large economic shocks, such as rare disasters on approximation errors, are largely unexplored.

Our formulation of DSGE models does not suffer from such limitations. First, we use closed-form solutions for reasonable parametric restrictions to study the determinants of the risk premium analytically. Second, we make use of powerful numerical methods to examine the properties of the risk premium for a broader parameter range without relying on local approximations (cf. Posch and Trimborn, 2010). We obtain optimal consumption, optimal hours and the risk premium as functions of financial wealth in the neoclassical production economy, while our closed-form solutions can be used to gauge and ensure the accuracy of the numerical method for large economic shocks. Thus we propose this formulation as a workable paradigm in the macro-finance literature.

This paper is closely related to Lettau (2003), who derives asset pricing implications in a real business cycle model using log-linear approximations. The present paper shows that the researcher overlooks potentially important properties of the risk premium implied by the neoclassical production economy when following this approach: a log-linear approximation of the consumption function, by construction, implies a constant risk premium. As we show in Section 3.2, this property is in fact obtained for knife-edge solutions only.

Our finding about the importance of the curvature of the consumption function for the risk premium in production economies relates to Jermann (2010), who studies the properties of the risk premium as implied by producers' first-order conditions. The author identifies the curvature of adjustment costs as a key determinant of the risk premium.

There is a literature documenting that the Barro-Rietz rare disaster hypothesis generates a sizable risk premium.[7] The most fundamental critique, however, is on the calibration of rare disasters. Although there is empirical evidence that economic disasters have been sufficiently frequent and large enough to make the hypothesis viable (cf. Barro, 2006), we emphasize that our results do not crucially depend on the rare disaster hypothesis. However, the hypothesis is excellent at illustrating the implications of the neoclassical production function (which for small risk premia would be negligible) for two reasons. First, it substantially increases the level of the risk premium without loosing analytical tractability. Second, it does not require other forms of non-linearities such as habit formation or recursive preferences which allows us to obtain very sharp results. Thus we do not contribute to the debate of why the historic equity premium seems too high given the low aggregate consumption volatility and our priors about risk aversion. In contrast, we confirm that the ability to buffer risk makes

[7] As a viable explanation for several macro-finance puzzles Gabaix (2008) and Wachter (2009) suggest variable intensity versions together with recursive preferences. This not only generates a time-varying risk premium but also increases the level of the premium. A critical view is found in Julliard and Gosh (2008).

it even more challenging to generate sizable risk premia in production economies.

The remainder of the paper is organized as follows. Section 2 solves a version of Lucas' fruit-tree model with exogenous, stochastic production and shows how to obtain the risk premium from the Euler equation without explicitly studying asset prices. Section 3 studies the effects of non-linearities on the risk premium in Merton's neoclassical growth model (our main economic insights are developed in Section 3.2). Section 4 concludes.

2 An endowment economy

This section illustrates our general equilibrium approach of computing the risk premium in an endowment economy. Our objective is twofold. First, we show that the risk premium can be obtained from the Euler equation without studying asset pricing implications. Second, we clarify the link between the risk premium and the premium of a market portfolio over the riskless asset in the presence of default risk and relate our results to the literature.

2.1 Lucas' fruit-tree model with rare disasters

Consider a fruit-tree economy (one risky asset or equity), and a riskless asset with default risk (government bond) similar to Barro (2006). One mechanism for partial default is the depreciation of the real value of nominal debt through high inflation.

2.1.1 Description of the economy

Technology. Consider an endowment economy (Lucas, 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time, $Y_t = A_t$ where A_t is the stochastic technology. Output is perishable. The law of motion of A_t will be taken to follow a Markov process,

$$dA_t = \mu_A A_t dt + \sigma_A A_t dB_t + (e^{\nu_A} - 1)A_{t-} dN_t. \tag{1}$$

B_t is a standard Brownian motion, N_t is a Poisson process at arrival rate λ, whereas μ_A and σ_A determine the instantaneous mean and variance of percentage changes in output in times without jumps. The jump size is assumed to be a constant fraction, $e^{\nu_A} - 1$, of output an instant before the jump, A_{t-}, ensuring that A_t does not jump to a negative value.

In this economy the bonds with default risk are issued exogenously by the government. Suppose that the price of the government bill follows

$$dp_0(t) = p_0(t) r dt + p_0(t_-) D_t dN_t, \tag{2}$$

where D_t is a random variable denoting a random default risk in case of a disaster, and q is the probability of default. For illustration, we assume

$$D_t = \begin{cases} 0 & \text{with} \quad 1-q \\ e^{\kappa}-1 & \text{with} \quad q \end{cases}.$$

Ownership of fruit-trees is determined at each instant in a competitive stock market, and the production unit has one outstanding perfectly divisible equity share. A share entitles its owner to all of the unit's instantaneous output in t. Shares are traded only at a competitively determined price, p_t. Suppose that the price of the risky asset follows

$$dp_t = \mu p_t dt + \sigma p_t dB_t + p_{t-} J_t dN_t, \tag{3}$$

where J_t is a random variable denoting the jump risk. Because prices fully reflect all available information, r, μ, σ and J_t will be determined later in general equilibrium.

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes expected life-time utility discounted at the subjective rate of time preference $\rho > 0$,

$$E \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \ u'' < 0.$$

Assuming no dividend payments, defining W_t as real financial wealth, and letting w_t denote the fraction of wealth held in the risky asset, the budget constraint reads (cf. appendix)

$$\begin{aligned} dW_t &= ((\mu - r) w_t W_t + r W_t - C_t)\, dt + w_t \sigma W_t dB_t + ((J_t - D_t) w_{t-} + D_t) W_{t-} dN_t \\ &= (\mu_M W_t - C_t)\, dt + \sigma_M W_t dB_t - \zeta_M(t_-) W_{t-} dN_t, \end{aligned} \tag{4}$$

in which we define

$$\mu_M \equiv (\mu - r) w_t + r, \quad \sigma_M \equiv w_t \sigma, \quad \zeta_M(t) \equiv (D_t - J_t) w_t - D_t.$$

$\zeta_M(t)$ is the exogenous stochastic jump-size. With probability q it takes the value ζ_M, and with probability $1-q$ the jump size is ζ_M^0. One can think of the original problem as having been reduced to a simple Ramsey problem of optimal consumption given that income is generated by the uncertain yield of a (composite) asset (cf. Merton, 1973).

Equilibrium properties. In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$. Outstanding shares will be held by capital owners. Our objective is to relate exogenous productivity changes to endogenous movements in asset prices, in particular we set out to study the determinants of the risk premium.

2.1.2 The Bellman equation and first-order condition

This section uses Bellman's idea to solve the control problem for the representative consumer and obtains the first-order conditions. Define the *value function* as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t) dt, \quad s.t. \quad (4), \quad W_0 > 0. \tag{5}$$

Choosing the control $C_s \in \mathbb{R}_+$ at time s, the Bellman equation reads

$$\begin{aligned} \rho V(W_s) \;=\; & \max_{C_s} \big\{ u(C_s) + (\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \\ & +(V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda \big\}. \end{aligned} \tag{6}$$

Because it is a necessary condition, the *first-order condition* reads

$$V_W(W_s) = u'(C_s) \tag{7}$$

for any interior solution at any time $s = t \in [0, \infty)$.

2.1.3 The Euler equation and the implied risk premium

Using the first-order condition (7), the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) \;=\; & \big((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t u''(C_t) C_W - u'(C((1-\zeta_M)W_t))(1-\zeta_M)q\lambda \\ & -u'(C((1-\zeta_M^0)W_t))(1-\zeta_M^0)(1-q)\lambda\big) dt \\ & +\pi u'(C_t) dB_t + (u'(C((1-\zeta_M(t_-))W_{t-})) - u'(C(W_{t-}))) dN_t, \end{aligned} \tag{8}$$

which implicitly gives the optimal consumption path, where $\pi \equiv \sigma_M W_t u''(C_t) C_W / u'(C_t)$ defines the market price of diffusion risk. Moreover, we define C_W as the marginal propensity to consume out of wealth, i.e., the slope of the *consumption function*. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$).

To shed light on the effects of uncertainty, we follow a similar approach as in Steger (2005). First, we multiply the Euler equation (8) by $1/u'(C_t)$. Second, we apply the expectation operator and finally collect terms to obtain the following optimality condition,

$$\underbrace{\rho - \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right]}_{\text{cost of forgone consumption}} \;=\; \underbrace{\mu_M - E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 + \frac{u'(C((1-\zeta_M(t))W_t))}{u'(C(W_t))}\zeta_M(t)\lambda\right]}_{\text{certainty equivalent rate of return}}.$$

In equilibrium, the *certainty equivalent rate of return*, i.e., the expected rate of return on saving in times without jumps, μ_M, less the expected value of the *risk premium*,

$$RP_t \;\equiv\; -\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 + E\left[\frac{u'(C((1-\zeta_M(t))W_t))}{u'(C(W_t))}\zeta_M(t)\lambda\right], \tag{9}$$

equals expected *cost of forgone consumption*, i.e., the subjective rate of time preference, and the expected rate of change of marginal utility. It denotes the percentage spread between the certainty equivalent rate of return (or shadow risk-free rate) and the expected rate of return of the market portfolio in times without jumps.

2.2 General equilibrium prices

This section shows that general equilibrium conditions pin down the prices in the economy. From the Euler equation (8), we obtain

$$\begin{aligned} dC_t &= \big((\rho-\mu_M+\lambda)u'(C_t)/u''(C_t)-\sigma_M^2 W_t C_W - \tfrac{1}{2}\sigma_M^2 W_t^2 C_W^2 u'''(C_t)/u''(C_t) \\ &\quad -E\left[u'(C((1-\zeta_M(t))W_t))(1-\zeta_M(t))\right]\lambda/u''(C_t)\big)dt \\ &\quad +\sigma_M W_t C_W dB_t + (C((1-\zeta_M(t))W_{t-})-C(W_{t-}))dN_t, \end{aligned} \tag{10}$$

where we employed the inverse function $g(u'(C_t)) \equiv C_t$ which has

$$g'(u'(C_t)) = 1/u''(C_t), \quad g''(u'(C_t)) = -u'''(C_t)/(u''(C_t))^3.$$

Economically, the assumption of concave utility, $u' > 0$, $u'' < 0$, implies risk aversion, while convex marginal utility, $u''' > 0$, implies a positive precautionary savings motive. Accordingly, $-u''/u'$ measures absolute risk aversion, whereas $-u'''/u''$ measures the degree of absolute prudence, i.e., the intensity of the precautionary savings motive (Kimball, 1990).

Because output is perishable, using the market clearing condition $Y_t = C_t = A_t$ gives

$$dC_t = \mu_A C_t dt + \sigma_A C_t dB_t + (e^{\nu_A}-1)C_{t-}dN_t. \tag{11}$$

Thus, the general equilibrium approach pins down asset prices as follows. Defining optimal jump in consumption as $\tilde{C}(W_t) \equiv C((1-\zeta_M(t))W_t)/C(W_t)$, market clearing requires the percentage jump in aggregate consumption to match the disaster size, $e^{\nu_A}-1 = \tilde{C}(W_t)-1$, which implies a constant jump term. For example, if consumption is linearly homogeneous in wealth (as shown for CRRA preferences below), the jump of the asset price satisfies[8]

$$C((1-\zeta_M(t))W_{t-})/C(W_{t-}) = 1-\zeta_M(t) \quad \Rightarrow \quad \zeta_M = \zeta_M^o = 1-e^{\nu_A}. \tag{12}$$

Similarly, the market clearing condition pins down

$$\mu_M - r = -\frac{u''(C_t)C_t^2}{u'(C_t)C_W W_t}\sigma_A^2 - \frac{u'(e^{\nu_A}C(W_t))}{u'(C(W_t))}\left((1-e^{\kappa})q+e^{\nu_A}-1\right)\lambda, \quad \sigma_M = \frac{C_t}{C_W W_t}\sigma_A, \tag{13}$$

[8] Conditioning on no default, $(\zeta_M(t)|D_t = 0) = \zeta_M^o$ gives $e^{\nu_A}-1 = -\zeta_M^o$, whereas conditioning on default, $(\zeta_M(t)|D_t = e^{\kappa}-1) = \zeta_M$ demands $e^{\nu_A}-1 = -\zeta_M$.

in which

$$r = \rho - \frac{u''(C_t)C_t}{u'(C_t)}\mu_A - \tfrac{1}{2}\frac{u'''(C_t)C_t^2}{u'(C_t)}\sigma_A^2 + \lambda - (1-(1-e^{\kappa})q)\,\frac{u'(e^{\nu_A}C_t)}{u'(C_t)}\lambda. \tag{14}$$

As a result, the higher the subjective rate of time preference, ρ, the higher is the general equilibrium interest rate which induces individuals to defer consumption (cf. Breeden, 1986). For convex marginal utility (decreasing absolute risk aversion), $u''' > 0$, a lower conditional variance of dividend growth, σ_A^2, a higher conditional mean of dividend growth, μ_A, and a higher default probability, q, decrease the bond price and increase the interest rate.

2.3 Components of the risk premium

Observe that the risk premium (9) in general equilibrium simplifies to

$$RP_t \;=\; \underbrace{-\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2}_{\text{diffusion risk}} + \underbrace{\frac{u'(e^{\nu_A}C(W_t))}{u'(C(W_t))}\zeta_M\lambda}_{\text{total jump risk}}, \tag{15}$$

whereas the market premium from (13), i.e., the premium of the expected rate of return on the market portfolio conditional on no disasters, in general equilibrium reads

$$\begin{aligned} \mu_M - r \;&=\; \underbrace{-\frac{u''(C_t)C_W W_t}{u'(C(W_t))}\sigma_M^2}_{\text{diffusion risk}} + \underbrace{(\zeta_M - (1-e^{\kappa})q)\,\frac{u'(e^{\nu_A}C(W_t))}{u'(C(W_t))}\lambda}_{\text{disaster risk}} \\ &=\; \underbrace{-\frac{u''(C_t)C_W W_t}{u'(C(W_t))}\sigma_M^2}_{\text{diffusion risk}} + \underbrace{\frac{u'(e^{\nu_A}C(W_s))}{u'(C(W_t))}\zeta_M\lambda}_{\text{total jump risk}} - \underbrace{(1-e^{\kappa})\,q\frac{u'(e^{\nu_A}C(W_t))}{u'(C(W_t))}\lambda}_{\text{default risk}}. \end{aligned} \tag{16}$$

Note that one would expect $\nu_A < 0$ and $\kappa < 0$ for a 'disaster' hypothesis.

In the presence of default risk, the market premium differs from the risk premium. The obvious reason is that we obtain the risk premium from the certainty equivalent rate of return. In contrast, the government bill has a risk of default which is not rewarded in the market premium, but it is reflected in the risk premium. If there was no default risk, the risk premium would have the usual interpretation of the market premium.

2.4 Analytical results

As shown in Merton (1971), the standard dynamic consumption and portfolio selection problem has explicit solutions where consumption is a linear function of wealth. For later reference, we provide the solution for constant relative risk aversion (CRRA).

Proposition 2.1 (CRRA preferences) *If utility exhibits constant relative risk aversion, i.e.,* $u(C_t) = C_t^{1-\theta}/(1-\theta)$, *optimal consumption is linear in wealth,* $C_t = C(W_t) = bW_t$, *where the marginal propensity to consume out of wealth is*

$$b \equiv \left(\rho + \lambda - (1-\theta)\mu_M - (1-\zeta_M)^{1-\theta}\lambda + (1-\theta)\theta\tfrac{1}{2}\sigma_M^2\right)/\theta.$$

The effective risk aversion is constant, $-V_{WW}(W_t)W_t/V_W(W_t) = \theta$.

Proof. see Appendix A.1.3 ■

Corollary 2.2 *Use the consumption function,* $C_t = C(W_t) = bW_t$, *and (16) to obtain*

$$\mu_M - r = \theta\sigma_M^2 + e^{-\theta\nu_A}\zeta_M\lambda - e^{-\theta\nu_A}\left(1-e^{\kappa}\right)q\lambda. \tag{17}$$

Similarly, the general equilibrium conditions pin down the jump size in (12), $\zeta_M = 1 - e^{\nu_A}$, *the variance in (13),* $\sigma_M = \sigma_A$, *and the face rate of the government bill in (14) as*

$$r = \rho + \theta\mu_A - \tfrac{1}{2}\theta(1+\theta)\sigma_A^2 + \lambda - \left(1-(1-e^{\kappa})q\right)e^{-\theta\nu_A}\lambda.$$

Corollary 2.3 *Use the consumption function,* $C_t = C(W_t) = bW_t$, *and the risk premium in general equilibrium (15), to obtain*

$$RP = \underbrace{\theta\sigma_A^2}_{\text{diffusion risk}} + \underbrace{e^{-\theta\nu_A}(1-e^{\nu_A})\lambda}_{\text{total jump risk}} = \mu_M - r + \underbrace{e^{-\theta\nu_A}(1-e^{\kappa})q\lambda}_{\text{default risk}}. \tag{18}$$

Hence, the risk premium indeed refers to the rewards that investors demand for bearing the systematic market risk over a riskless alternative (or the 'shadow' risk-free rate in cases such a riskless alternative does not exist). Two results are important. Our first result is that the risk premium can be obtained directly from the Euler equation in (9) without studying asset prices (as in Section 2.2). Second, the risk premium in (18) is constant and its determinants are (effective) risk aversion and consumption volatility.

Similar to Barro (2006), the premium for diffusion risk $\theta\sigma_A^2$ increases by the total jump risk $e^{-\theta\nu_A}(1-e^{\nu_A})\lambda$. The intuition of why rare events may solve the risk premium puzzle is as follows. Even for logarithmic utility, $\theta = 1$, and for low-probability events, $\lambda = 1\%$, the premium for the jump risk in percentage points, $e^{-\nu_A} - 1$, can be very large. For the case of 'disasters' one would expect ν_A to be negative. The more negative the parameter, the more severe is the disaster and $\nu_A \to -\infty$ denotes complete destruction.

As we show below, the reason why the risk premium is constant is that the consumption function is homogeneous (of degree $k = 1$), which implies that the effective risk aversion is constant. A time-varying disaster size and/or arrival rate (i.e., stochastic volatility) would imply both a time-varying and a larger risk premium (cf. Gabaix, 2008; Wachter, 2009).

3 A neoclassical production economy

This section illustrates that non-linearities in a neoclassical DSGE model imply interesting asset market implications. In particular they can generate a time-varying and asymmetric risk premium. We use a version of Merton's (1975) asymptotic theory of growth under uncertainty and compare our results to the reference case of an endowment economy.

3.1 A model of growth under uncertainty

This section assumes that there is no riskless asset. We use the certainty equivalent rate of return (or shadow risk-free rate) to obtain the risk premium from the Euler equation.

3.1.1 Description of the economy

Technology. At any time, the economy has some amounts of capital, hours, and knowledge, and these are combined to produce output. The production function is a constant return to scale technology $Y_t = A_t F(K_t, H_t)$, where K_t is the aggregate capital stock, H_t is hours worked as a fraction of total hours, and A_t is the stock of knowledge or total factor productivity (TFP), which in turn is driven by a standard Brownian motion B_t,

$$dA_t = \mu_A A_t dt + \sigma_A A_t dB_t. \tag{19}$$

In contrast to the endowment economy ownership of A_t is not determined in a competitive stock market. Each firm has free access to the stock of knowledge.

The physical capital stock increases if gross investment exceeds capital depreciation,

$$dK_t = (I_t - \delta K_t)dt + \sigma_K K_t dZ_t + (e^{\nu_K} - 1)K_{t-} dN_t, \tag{20}$$

where Z_t is a standard Brownian motion (uncorrelated with B_t), and N_t is a standard Poisson process with arrival rate λ. Unlike in Merton's (1975) model, the assumption of stochastic depreciation introduces instantaneous riskiness indeed making physical capital a risky asset. The fundamental difference to Lucas' endowment economy is that the outstanding equity shares follow a stochastic process, i.e., the number of trees is stochastic. An endowment economy would be obtained for $F(K_t, H_t) \equiv 1$.

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer seeks to maximize

$$E_0 \int_0^\infty e^{-\rho t} u(C_t, H_t) dt, \quad u_C > 0,\ u_H \leq 0,\ u_{CC} \leq 0,\ u_{CC}u_{HH} - (u_{CH})^2 \geq 0, \tag{21}$$

subject to

$$dW_t = ((r_t - \delta)W_t + H_t w_t^H - C_t)dt + \sigma_K W_t dZ_t + (e^{\nu_K} - 1)W_{t-} dN_t. \tag{22}$$

$W_t \equiv K_t$ denotes individual wealth, r_t is the rental rate of capital, and $H_t w_t^H$ is labor income. The paths of factor rewards are taken as given by the representative consumer. For later reference, we define the savings rate $s(r_t, w_t^H, C_t, H_t, W_t) \equiv (1 - C_t)/(r_t W_t + H_t w_t^H)$.

Equilibrium properties. In equilibrium, factors of production are rewarded with value marginal products, $r_t = Y_K$ and $w_t^H = Y_H$. The goods market clearing condition demands

$$Y_t = C_t + I_t. \tag{23}$$

Solving the model requires the aggregate accumulation constraints (19) and (20), the goods market equilibrium (23), equilibrium factor rewards of competitive firms, and the first-order condition for consumption and hours. It gives a system of equations which, given initial conditions, determines the paths of K_t, Y_t, r_t, w_t^H, C_t and H_t, respectively.

3.1.2 The Bellman equation and first-order conditions

Define the value function as

$$V(W_0, A_0) = \max_{\{C_t, H_t\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t, H_t) dt \quad s.t. \quad (22),\ (19), \quad W_0, A_0 > 0, \tag{24}$$

denoting the present value of expected utility along the optimal program. As shown in the appendix, the *Bellman equation* for this problem reads

$$\begin{aligned}\rho V(W_s, A_s) = \max_{C_s, H_s} \big\{ & u(C_s, H_s) + ((r_s - \delta) W_s + H_s w_s^H - C_s) V_W \\ & + \tfrac{1}{2} \left(V_{AA} \sigma_A^2 A_s^2 + V_{WW} \sigma_K^2 W_s^2 \right) + V_A \mu_A A_s + [V(e^{\nu_K} W_s, A_s) - V(W_s, A_s)] \lambda \big\}\end{aligned}$$

for any $s \in [0, \infty)$. Hence, the *first-order conditions* for any interior solution are

$$u_C(C_t, H_t) = V_W(W_t, A_t), \tag{25}$$

$$-u_H(C_t, H_t) = w_t^H V_W(W_t, A_t), \tag{26}$$

for any $t \in [0, \infty)$, making optimal consumption and hours functions of the state variables $C_t = C(W_t, A_t)$ and $H_t = H(W_t, A_t)$, respectively. Both intertemporal and intra-temporal first-order conditions (25) and (26) pin down the opportunity cost of leisure,

$$w_t^H = -\frac{u_H(C_t, H_t)}{u_C(C_t, H_t)}, \quad u_H \neq 0. \tag{27}$$

For utility functions with $u_H = 0$, the household optimally supplies total hours to production, $H = 1$. In this case the optimality condition reduces to (25). Allowing the labor-leisure choice to be endogenous, however, has potentially important asset pricing implications.[9]

[9] Bodie, Merton and Samuelson (1992) show that the individual's human capital, which essentially is the same as a financial asset except that it is not traded, is valued by the individual as if it were a traded asset.

3.1.3 The Euler equation and the implied risk premium

After some algebra the *Euler equation* for consumption is (cf. appendix)

$$
\begin{aligned}
du_C &= (\rho-(r_t-\delta)+\lambda)u_C dt - u_C(C(e^{\nu_K}W_t,A_t),H(e^{\nu_K}W_t,A_t))e^{\nu_K}\lambda dt \\
&\quad -\sigma_K^2\left(u_{CC}(C_t,H_t)C_W+u_{CH}(C_t,H_t)H_W\right)W_t dt \\
&\quad +(C_A A_t\sigma_A dB_t + C_W W_t\sigma_K dZ_t)u_{CC}+(H_A A_t\sigma_A dB_t+H_W W_t\sigma_K dZ_t)u_{CH} \\
&\quad +\left[\frac{u_C(C(e^{\nu_K}W_{t-},A_{t-}),H(e^{\nu_K}W_{t-},A_{t-}))}{u_C(C(W_{t-},A_{t-}),H(W_{t-},A_{t-}))}-1\right]u_C(C_{t-},H_{t-})dN_t, \qquad (28)
\end{aligned}
$$

which implicitly determines the optimal consumption path. In order to shed some light on the effects of uncertainty in this economy, we rewrite the Euler equation and obtain

$$
\begin{aligned}
\rho-\frac{1}{dt}E\left[\frac{du_C(C_t,H_t)}{u_C(C_t,H_t)}\right] &= E(r_t-\delta)-E\left[-\frac{u_{CC}(C_t,H_t)C_W+u_{CH}(C_t,H_t)H_W}{u_C(C_t,H_t)}W_t\sigma_K^2\right] \\
&\quad -E\left[\frac{u_C(C(e^{\nu_K}W_t,A_t),H(e^{\nu_K}W_t,A_t))}{u_C(C(W_t,A_t),H(W_t,A_t))}(1-e^{\nu_K})\lambda\right].
\end{aligned}
$$

Similar to the endowment economy, the left-hand side equals the cost of forgone consumption, which in equilibrium equals the certainty equivalent return on saving on the right-hand side, i.e., the expected net return on assets minus the *risk premium*,

$$
RP_t \equiv -\frac{u_{CC}C_W+u_{CH}H_W}{u_C(C_t,H_t)}W_t\sigma_K^2+\frac{u_C(C(e^{\nu_K}W_t,A_t),H(e^{\nu_K}W_t,A_t))}{u_C(C(W_t,A_t),H(W_t,A_t))}(1-e^{\nu_K})\lambda. \qquad (29)
$$

It is remarkable that the structure is equivalent to the endowment economy (15). The most obvious result is that the risk premium indeed refers to the rewards that investors demand for bearing the systematic risk, while it does not directly account for the risk of a stochastically changing total factor productivity (19).

3.2 Analytical results

In order to shed light on the properties of the risk premium (29) in DSGE models, this section makes specific assumptions about the functional forms for the production side and household's preferences. In what follows, we consider the class of utility functions which exhibits CRRA with respect to gambles in both consumption and leisure,

$$
u(C_t,H_t)=\frac{(C_t(1-H_t)^{\psi})^{1-\theta}}{1-\theta}, \quad \theta>0,\ \psi\geq 0. \qquad (30)
$$

Similar to Turnovsky and Smith (2006), the parameter ψ measures the preference for leisure. To ensure concavity, we restrict $\theta-(1-\theta)\psi\geq 0$ which is the consumption-leisure-based measure of relative risk aversion (cf. Swanson, 2010). For the case of $\psi=0$, the utility function (30) reduces to the standard CRRA utility framework.

As a result, the risk premium in the production economy in (29) reads

$$\begin{aligned} RP_t &= \theta\frac{C_W(W_t,A_t)}{C(W_t,A_t)}W_t\sigma_K^2-(1-\theta)\frac{H_W(W_t,A_t)}{1-H(W_t,A_t)}W_t\sigma_K^2 \\ &\quad+\frac{C(e^{\nu_K}W_t,A_t)^{-\theta}(1-H(e^{\nu_K}W_t,A_t))^{\psi(1-\theta)}}{C(W_t,A_t)^{-\theta}(1-H(W_t,A_t))^{\psi(1-\theta)}}(1-e^{\nu_K})\lambda. \end{aligned} \tag{31}$$

Below we show that both the consumption function $C_t = C(A_t,W_t)$ and optimal hours $H_t = H(A_t,W_t)$ as a function of the state variables are available in closed form for parametric restrictions. This in turn gives closed-form expressions for the risk premium in (31).

Proposition 3.1 (linear-consumption-rule) *Suppose the production function is of the type Cobb-Douglas, $Y_t = A_tK_t^{\alpha}H_t^{1-\alpha}$ and there is no disutility from labor supply, $\psi = 0$. For the case where $\alpha = \theta$, optimal consumption is linear in wealth,*

$$C_t = C(W_t) = \phi W_t, \quad H = 1, \tag{32}$$
$$\text{where} \quad \phi \equiv (\rho-(e^{(1-\theta)\nu_K}-1)\lambda+(1-\theta)\delta)/\theta+\tfrac{1}{2}(1-\theta)\sigma_K^2.$$

The effective risk aversion is constant, $-V_{WW}(W_t)W_t/V_W(W_t) = \theta$.

Proof. see Appendix A.2.2 ■

Corollary 3.2 *Using the consumption function and optimal hours in (32) and (31),*

$$RP = \theta\sigma_K^2 + e^{-\theta\nu_K}(1-e^{\nu_K})\lambda. \tag{33}$$

Proposition 3.3 (constant-saving-function) *Suppose the production function is of the type Cobb-Douglas, $Y_t = A_tK_t^{\alpha}H_t^{1-\alpha}$. The knife-edge value*

$$\bar\rho \equiv (e^{\nu_K(1-\alpha\theta)}-1)\lambda-(1-\alpha\theta)\delta-\theta\mu_A+\tfrac{1}{2}\left(\theta(1+\theta)\sigma_A^2-\alpha\theta(1-\alpha\theta)\sigma_K^2\right), \tag{34}$$

determines the shape of optimal hours as a function of the state variables. For the case where $\rho = \bar\rho$, optimal consumption is proportional to income, and optimal hours are constant,

$$C_t = \frac{\theta-1}{\theta}A_tW_t^{\alpha}H^{1-\alpha}, \quad H = \frac{\theta(1-\alpha)}{\theta(1-\alpha)-\psi(1-\theta)} \quad \theta > 1. \tag{35}$$

The effective risk aversion is constant, $-V_{WW}(W_t)W_t/V_W(W_t) = \alpha\theta$.

Proof. see Appendix A.2.3 ■

Corollary 3.4 *Using the consumption function and optimal hours in (35) and (31),*

$$RP = \alpha\theta\sigma_K^2 + e^{-\alpha\theta\nu_K}(1-e^{\nu_K})\lambda. \tag{36}$$

The closed-form expressions for the risk premium in (33) and (36) hold our main results.

3.2.1 Discussion of the analytical results

Our first result is that the risk premium in DSGE models depends on the curvature of the consumption function. To make our point more explicit, suppose that labor supply is constant (either for $\psi = 0$ or as shown above $\rho = \bar{\rho}$). In fact, any homogenous function, where $C_W(W_t, A_t)W_t = kC(W_t, A_t)$ or equivalently $C(cW_t, A_t) = c^k C(W_t, A_t)$ for $c, k \in \mathbb{R}_+$, implies a constant risk premium in (29). Technically, the consumption function needs to be homogenous of degree k in wealth. This leads us to our second result because we are now in a position to understand why the risk premium in DSGE models generally has a time-varying property. Because homogenous functions are obtained only for knife-edge restrictions, the risk premium generally will be dependent on wealth which in turn implies a time-varying behavior and predictability since wealth is changing stochastically. The bottom line is that the non-linearity of the production function generally implies non-homogeneous consumption functions and thus generates a time-varying risk premium.

Economically, the reason why the risk premium depends on the curvature of the consumption function (and can vary over time) is that the optimal response to disasters or shocks will depend on the level of wealth. An individual with high levels of financial wealth will adjust his or her optimal consumption differently than an individual with no financial wealth. Though the direct utility function has CRRA with respect to consumption, the indirect utility function (i.e., the value function) does *not* exhibit CRRA with respect to wealth except for the knife-edge cases above. This finding is closely related to the link between the marginal propensity to consume and the effective risk aversion of the value function: a higher marginal propensity to consume out of gross wealth (inclusive of labor income) raises the effective risk aversion by raising the consumption covariance from a given financial risk (Carroll and Kimball, 1996, p.982).[10] In other words, if the marginal propensity to consume falls more quickly with wealth than it would for a constant savings rate, then the effective risk aversion of the value function will be lower at higher levels of wealth.

Given our discussion above, we are now in a position to understand why the technique of log-linearizing all relevant equations following Campbell (1994) implies a constant risk premium (cf. Lettau, 2003). In fact, this approach gives the endogenous variables (in logs) as a linear function of the state variables (in logs) which makes the consumption function homogeneous in the state variables. So what are we missing by using a log-linear solution? As discussed above, a homogeneous consumption function implies a constant risk premium. However, the risk premium implied by the exact solution, except in our knife-edge cases,

[10] Our consumption function refers to financial wealth (exclusive of labor income), while previous work refers to gross wealth or total wealth (i.e., financial and human wealth). Carroll and Kimball's result is based on a partial equilibrium model leaving the processes for labor income and capital returns exogenous.

exhibits time-varying and asymmetric behavior due to changes in effective risk aversion. Thus by using log-linear approximations the researcher overlooks potentially important properties of the risk premium implied by the neoclassical production economy.

Unfortunately, an analytical study of the risk premium without parametric restrictions is not possible. Though clearly being knife-edge cases, our explicit solutions are important in order to understand the determinants of the risk premium in DSGE models. Both analytical solutions imply that the consumption function is homogenous in wealth and optimal hours are constant, which in turn implies a constant risk premium (cf. Figure 1).[11] Below we study the implications of allowing the parameters to take different values.

3.3 Numerical results

This section uses numerical solutions in order to illustrate our third result on the asymmetry of the risk-premium over the business cycle for the case of a non-linear production function. For illustration, we consider the case where $\sigma_K = \sigma_A = \mu_A = 0$ and the production is of the type Cobb-Douglas, $Y_t = AK_t^{\alpha} H_t^{1-\alpha}$ with $A = 1$. The parametric assumptions do not affect our conclusions but substantially reduce the computational burden.

3.3.1 The risk premium with inelastic labor supply ($\psi = 0$)

Consider the case of inelastic labor supply by assuming $u_H = 0$, which eases interpretation and simplifies notation. We refer to the numerical results where $u_H \neq 0$ in Section 3.3.3.

From the Euler equation (28), the reduced form representing the dynamics of the DSGE model can be summarized by a system of two stochastic differential equations (SDEs) as

$$\begin{aligned} dW_t &= ((r_t - \delta)W_t + w_t^H - C_t)dt - (1 - e^{\nu_K})W_{t-}dN_t, \\ dC_t &= -\frac{u_C(C_t)}{u_{CC}(C_t)}(r_t - \delta - \rho - \lambda)dt - \frac{u_C(C(e^{\nu_K}W_t))}{u_{CC}(C(W_t))}e^{\nu_K}\lambda dt + [C(e^{\nu_K}W_{t-}) - C(W_{t-})]dN_t, \end{aligned}$$

where $r_t = Y_K$ and $w_t^H = Y_H$, augmented by boundary conditions for the beginning and the end of the time horizon. This problem can be solved using the Waveform relaxation algorithm proposed in Posch and Trimborn (2010).

For later reference, because $u_H = 0$ (in particular $\psi = 0$), we are studying the case of CRRA preferences with relative risk aversion θ, and from (31) the risk premium reads

$$RP_t = \frac{C(e^{\nu_K}W_t)^{-\theta}}{C(W_t)^{-\theta}}(1 - e^{\nu_K})\lambda. \tag{37}$$

[11] For $\alpha = \theta$ and $\psi = 0$, the consumption function becomes a linear function in wealth, i.e., it is linearly homogeneous or homogeneous of degree one. In the case of $\rho = \bar{\rho}$, which is only possible for values $\theta > 1$, the consumption function becomes homogeneous of degree α.

The numerical solution to the non-linear system of stochastic differential equations is the consumption function, $C_t = C(W_t)$, which is obtained from the optimal paths of control and state variables computed for the complete state space $W_t \in \mathbb{R}_+$. In particular our procedure does not rely on local approximation methods, but directly solves the non-linear system globally (cf. Posch and Trimborn, 2010). According to (37), we obtain the risk premium by evaluating the consumption function at two points in the state space.

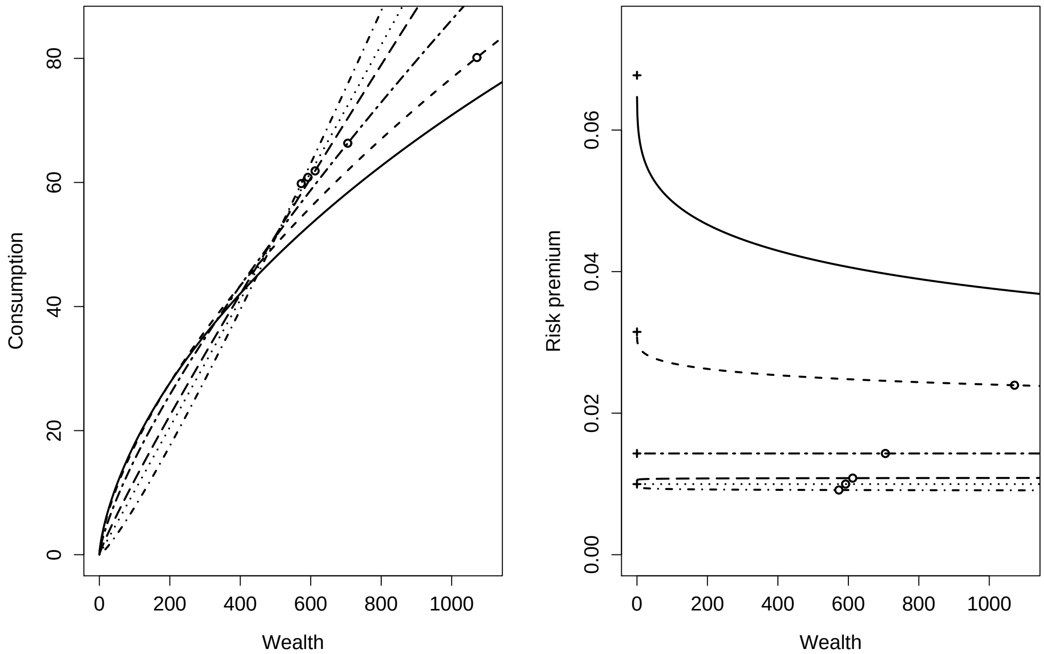
Figure 1 shows the optimal consumption function and the resulting risk premium (37) for different values for the parameter of relative risk aversion θ.

3.3.2 Discussion of the numerical results

For $\theta = \alpha$ the consumption function in Figure 1 is linearly homogenous (dotted) with slope ϕ which corresponds to the analytical solution in (32). In this singular case the risk premium is $e^{-\theta\nu_K}(1 - e^{\nu_K})\lambda$, which is equivalent to the endowment economy. At each point in time, the expected proportional change in marginal utility equals the expected change in capital rewards which implies a constant risk premium in (37). For $\theta < \alpha$ the consumption function is convex, and the marginal propensity to consume increases with wealth, $C(e^{\nu_K} W_t) < e^{\nu_K} C(W_t)$. This increase is less rapid than the increase of the consumption-wealth ratio, which lowers the effective risk aversion. Hence, the risk premium is convex and has the upper bound $e^{-\alpha\nu_K}(1-e^{\nu_K})\lambda$ for wealth approaching zero. For $\theta > \alpha$, which is the empirically most plausible scenario, the consumption function has the standard form, i.e., strictly concave and the marginal propensity to consume is decreasing with wealth, $C(e^{\nu_K} W_t) > e^{\nu_K} C(W_t)$. In this case, the properties of the risk premium (37) depend on whether the subjective rate of time preference ρ exceeds or falls short of the knife-edge value $\bar{\rho}$ in (34).

At the knife-edge value of $\rho = \bar{\rho}$ the consumption function is homogeneous of degree α which refers to the analytical solution in (35) with constant savings rate, $s = 1/\theta$, and the risk premium is $e^{-\alpha\theta\nu_K}(1 - e^{\nu_K})\lambda$. For $\rho < \bar{\rho}$ the individual prefers a higher savings rate, $s(W_t) > s$, and the marginal propensity to consume falls more quickly with wealth than it would for a constant savings rate (or consumption-income ratio) which lowers effective risk aversion at higher levels of wealth. Because $s(W_t)$ is increasing in wealth and bounded by unity, $s < s(W_t) < 1$, the risk premium is convex and has the upper bound $e^{-\alpha\theta\nu_K}(1 - e^{\nu_K})\lambda$ for wealth approaching zero. Similarly, for $\rho > \bar{\rho}$ the savings rate is smaller, $s(W_t) < s$, and the marginal propensity to consume falls less quickly in wealth than for a constant savings rate, which raises the effective risk aversion at higher levels of wealth. As $s(W_t)$ is decreasing in wealth, the risk premium in (37) is concave with lower bound $e^{-\theta\alpha\nu_K}(1 - e^{\nu_K})\lambda$ for sufficiently risk averse individuals, $\theta \geq 1$. Otherwise the substitution effect dominates the precautionary savings effect which depresses savings and increases the marginal propensity

Figure 1: Risk premia in a production economy



Notes: These figures illustrate optimal consumption (left panel) and the risk premium (right panel) as functions of wealth ($\circ$ are steady-state values) for different levels of relative risk aversion and $\sigma_K = \sigma_A = \mu_A = 0$. The calibration of other parameters is $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu_K}, \psi) = (0.05, 0.75, \cdot, 0.1, 0.017, 0.4, 0)$; where $\theta = 0.5$ (dotdash), $\theta = 0.75$ (dotted), $\theta = 1$ (longdash), $\theta = 1.9406$ (twodash) which refers to the knife-edge case $\rho = \bar{\rho}$ in (34), $\theta = 4$ (dashed), and $\theta = 6$ (solid).

to consume (Weil, 1990). Since the consumption function is concave for $\theta > \alpha$ due to the non-linear production function, effective risk aversion remains higher than for $\theta = \alpha$, such that the lower bound is $e^{-\max(\theta,1)\alpha\nu_K}(1 - e^{\nu_K})\lambda$ for wealth approaching zero.

In our numerical study $\bar{\rho}$ depends on the arrival rate, λ, the disaster size, $e^{\nu_K} - 1$, the output elasticity of capital, α, and the risk aversion, θ, which coincides with the inverse of the intertemporal elasticity of substitution (IES), and the rate of depreciation, δ. For the case $\alpha\theta > 1$, which is when the output elasticity of capital exceeds the IES, this critical value is positive, $\bar{\rho} > 0$, and vice versa. For the empirically most plausible calibrations, e.g., for $\alpha \approx 0.33$ and $\theta \approx 4$, we have $\alpha\theta > 1$ and obtain a positive knife-edge value, $\bar{\rho} > 0$.

Our numerical study of the risk premium yields our three results. For the empirically most plausible scenarios, we confirm our analytical result that (i) the risk premium depends on wealth and thus will be time-varying over the business cycle. In prosperous states of the economy with higher transitional growth rates (capital scarcity), the risk premium is higher than in periods with lower - or even negative - growth rates (capital abundance). In other words, after a disaster the risk premium jumps to a higher value as capital stock is destroyed, and then subsequently return to lower values as more capital is accumulated, which implies (ii) a predictable component (cf. Campbell and Cochrane, 1999). Finally, we find that the response of the risk premium is (iii) asymmetric: the increase after a negative shock is larger than the decrease after a positive shock with the same size.

Allowing for (Gaussian) stochastic depreciation, $\sigma_K > 0$, and/or a second state variable in the form of time-varying TFP, $\mu_A \neq 0$, $\sigma_A > 0$, the risk premium can be obtained from (31), and the same analysis could be conducted. The consumption function is concave in wealth for $\theta \geq \alpha$ and the risk premium, conditional on A_t, has the same properties as in Figure 1. However, there are three main differences. First, since the individual is willing to hedge against the diffusion risk (stochastic investment opportunities), the risk premium will be slightly higher.[12] Second, the risk premium in general also depends indirectly on TFP through the optimal consumption function, $C_t = C(K_t, A_t)$. Third, the knife-edge value $\bar{\rho}$ as from (34) decreases in the mean, μ_A, but increases in the variance σ_A^2 of TFP growth. For the case $\alpha\theta > 1$ it increases in the variance of stochastic depreciation, σ_K^2.

3.3.3 The role of elastic labor supply ($\psi > 0$)

This section allows for elastic labor supply in the neoclassical DSGE model. Our objective is to study how the properties of the risk premium is affected by the ability of individuals to buffer risk through their labor-leisure choice for $u_H \neq 0$, in particular $\psi > 0$.[13]

In the general case where $\psi > 0$, the reduced-form dynamics can be summarized by the budget constraint (22) and Euler equations for both consumption and hours. In fact, the condition (27) can be used to relate optimal consumption as a function of hours and state variables which reduces the dimensionality of the problem. Observe that (27) implies

$$1 - H(W_t) \quad = \quad \frac{\psi}{1-\alpha} \frac{C(W_t) H(W_t)^{\alpha}}{W_t^{\alpha}}. \tag{38}$$

Because (38) needs to hold for any levels of wealth, in particular for $e^{\nu_K} W_t$, this optimality condition pins down the optimal jump terms for consumption and hours as

$$\tilde{C}(W_t) \quad = \quad \frac{1 - \tilde{H}(W_t) H(W_t)}{1 - H(W_t)} \tilde{H}(W_t)^{-\alpha} e^{\alpha \nu_K}, \tag{39}$$

in which $\tilde{C}(W_t) \equiv C(e^{\nu_K} W_t)/C(W_t)$ and $\tilde{H}(W_t) \equiv H(e^{\nu_K} W_t)/H(W_t)$. Hence, $1 - \tilde{C}(W_t)$ denotes the percentage drop of optimal consumption after a disaster. As a result, we can neglect the Euler equation for consumption since technically (38) and (39) give consumption as functions of optimal hours and wealth, $C(W_t) = C(H(W_t), W_t)$ and $\tilde{C}(W_t) = \tilde{C}(H(W_t))$. Economically, optimal behavior of consumption is described completely by optimal hours and wealth through the intra-temporal optimality condition (27).

[12] Since the diffusion risk is of less importance, this effect is negligible (cf. Tables A.1 and A.2).

[13] It is well known that labor flexibility introduces an additional margin along which an individual can buffer risk (Turnovsky and Bianconi, 2005, p.325).

As shown in the appendix, for $0 < H_t < 1$ the reduced form can be summarized as

$$
\begin{aligned}
dH_t &= \frac{\rho - (1-\theta)r_t + (1-\alpha\theta)\delta + \lambda - \alpha\theta C_t/W_t}{\alpha\theta H_t^{-1} - (\psi - \theta\psi - \theta)(1-H_t)^{-1}} dt \\
&\quad - \frac{\tilde{C}(W_t)^{-\theta+(1-\theta)\psi} \tilde{H}(W_t)^{(1-\theta)\psi\alpha} e^{\nu_K - (1-\theta)\psi\alpha\nu_K} \lambda}{\alpha\theta H_t^{-1} - (\psi - \theta\psi - \theta)(1-H_t)^{-1}} dt + (H(e^{\nu_K} W_{t-}) - H(W_{t-}))dN_t, \\
dW_t &= ((r_t - \delta)W_t + H_t w_t^H - C_t)dt - (1 - e^{\nu_K})W_{t-}dN_t,
\end{aligned}
$$

which we use for our numerical results. Finally, the risk premium (31) is obtained from

$$
RP_t = \tilde{C}(W_t)^{(1-\theta)\psi-\theta} \tilde{H}(W_t)^{(1-\theta)\psi\alpha} e^{-(1-\theta)\psi\alpha\nu_K} (1 - e^{\nu_K})\lambda. \tag{40}
$$

Our analytical results suggest that even for $\psi > 0$ our insights about the properties of the risk premium do not change for the particular rate of time preference $\bar{\rho}$ in (34). Clearly, $\rho = \bar{\rho}$ is a knife-edge condition which ensures that optimal hours in (35), $H(W_t) = H$ and the savings rate $s = 1/\theta$ are constant, and the risk premium is $e^{-\alpha\theta\nu_K}(1 - e^{\nu_K})\lambda$. In this singular case, the parameter measuring the preference for leisure, ψ, does not affect the risk premium or the savings rate, though it affects the fraction of hours supplied to production. As it turns out, the qualitative properties of the risk premium are generally unaffected by the presence of elastic labor supply, though the quantitative results change slightly.

Our numerical results can be summarized by plotting the consumption function, optimal hours, and the risk premium as functions of financial wealth (cf. Figures A.1 and A.2). In what follows, we restrict our discussion to the empirically most relevant case where $\theta \geq 1$.

For $\rho < \bar{\rho}$ the individual prefers a higher savings rate, $s(W_t) > s$, and supplies more hours, $H(W_t) > H$. Both the consumption function and optimal hours are concave, while the risk premium is convex in wealth and has the upper bound $e^{-\alpha\theta\nu_K}(1 - e^{\nu_K})\lambda$ for wealth approaching zero. For $\rho > \bar{\rho}$ the savings rate is lower, $s(W_t) < s$, the individual supplies less hours, $H(W_t) < H$, and the risk premium is concave with lower bound $e^{-\theta\alpha\nu_K}(1 - e^{\nu_K})\lambda$.

An empirically testable implication is the correlation between hours and consumption. In the data, hours and consumption are positively correlated which in turn implies a negative correlation between consumption and leisure (cf. Lettau and Uhlig, 2000). We may infer this property directly from the consumption function and optimal hours. For $\rho = \bar{\rho}$ there is zero correlation, while for $\rho < \bar{\rho}$ consumption and hours are concave functions of financial wealth (or capital stock per effective worker). This implies a positive correlation since wealth is changing stochastically. It is only for $\rho > \bar{\rho}$ that optimal hours are convex in financial wealth. In turn this implies a counterfactual negative correlation for any concave consumption function. Thus, the empirically most plausible case $\rho < \bar{\rho}$ implies strictly concave functions for both consumption and hours as well as time-varying and asymmetric risk premia similar to the benchmark case of constant labor supply $\psi = 0$.

In summary, the extension to endogenous labor supply is able to generate empirically plausible correlations for consumption and leisure. Though our main results on the shape and the time-varying property of the risk premium are not affected, the ability to buffer risk through the labor-leisure choice makes it even more challenging to generate sizable risk premia in production economies. One interesting extension of this framework could therefore examine the role of other sources of non-linearities such as capital adjustment cost and/or habit formation which affect effective risk aversion and thus the risk premium.

4 Conclusion

In this paper we study how non-linearities affect asset pricing implications in a production economy. We derive closed-form solutions of the Lucas' fruit-tree model and compare the resulting risk premium to those obtained from a model with non-linearities in the form of a neoclassical production function. For this purpose, we formulate our DSGE models in continuous time to obtain analytical solutions, which are important knife-edge cases for numerical work. Our key result is that these non-linearities can generate time-varying, asymmetric risk premia and predictability over the business cycle. The economic intuition is that the individual's effective risk aversion, except for singular cases, is not constant in a neoclassical production economy. We show that non-normalities in the form of rare disasters substantially increase the economic relevance of these (empirical) key features.

From a methodological point of view, this paper shows that formulating the endowment economy or non-trivial production models in continuous time gives analytical solutions for reasonable parametric restrictions or functional forms. Analytical solutions are useful for macro-finance models for at least two reasons. First, they are points of reference from which numerical methods can be used to explore a broader class of models. Second, they shed light on asset market implications without relying purely on numerical methods. This circumvents problems induced by approximation schemes which could be detrimental when studying the effects of uncertainty. Along these lines, we propose the continuous-time formulation of DSGE models as a workable paradigm in macro-finance.

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A Appendix

A.1 Lucas' fruit-tree model with rare disasters

A.1.1 Deriving the budget constraint

Consider a portfolio strategy which holds n_t units of the risky asset and $n_0(t)$ units of the riskless asset with default risk, such that $W_t = n_0(t)p_0(t) + p_t n_t$ denotes the portfolio value.

Using Itô's formula, it follows that

$$\begin{aligned} dW_t &= p_0(t)dn_0(t) + n_0(t)p_0(t)rdt + p_tdn_t + w_t\mu W_tdt + w_t\sigma WdB_t \\ &\quad + (w_{t-}J_t + (1-w_{t-})D_t)\, W_{t-}dN_t, \end{aligned}$$

where $w_tW_t \equiv n_tp_t$ denotes the amount invested in the risky asset. Since investors use their savings to accumulate assets, assuming no dividend payments, $p_0(t)dn_0(t) + p_tdn_t = -C_tdt$,

$$dW_t = ((\mu - r)w_tW_t + rW_t - C_t)\, dt + \sigma w_tW_tdB_t + ((J_t - D_t)w_{t-} + D_t)\, W_{t-}dN_t.$$

where J_t is the jump size of the risky asset, which with probability q takes the value $e^{\nu_2} - 1$ (no default) and with probability $1-q$ the jump size is $e^{\nu_1} - 1$ (default).

A.1.2 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s) = \max_{(w_s,C_s)} \left\{ u(C_s) + \frac{1}{dt}E_sdV(W_s) \right\}. \tag{41}$$

Using Itô's formula (see e.g. Sennewald, 2007),

$$\begin{aligned} dV(W_s) &= \left((\mu_MW_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2W_s^2V_{WW}\right) dt + \sigma_MW_sV_WdB_t + (V(W_s) - V(W_{s-}))dN_t \\ &= \left((\mu_MW_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2W_s^2V_{WW}\right) dt + \sigma_MW_sV_WdB_t \\ &\quad +(V((1-\zeta_M(t_-))W_{s-}) - V(W_{s-}))dN_t, \end{aligned}$$

If we apply the expectation operator to the integral form, and use the property of stochastic integrals, we may write using $\zeta_M \equiv E(\zeta_M(t)|D_t = e^\kappa - 1) = 1 - e^\kappa - (e^{\nu_1} - e^\kappa)w$ (default) and $\zeta_M^0 \equiv E(\zeta_M(t)|D_t = 0) = (1 - e^{\nu_2})w$ (no default),

$$\begin{aligned} E_sdV(W_s) &= \big(\, (\mu_MW_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2W_s^2V_{WW} \\ &\quad +(V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda\big)dt. \end{aligned}$$

Inserting into (41) gives the Bellman equation

$$\begin{aligned} \rho V(W_s) &= \max_{(w_s,C_s)} \big\{u(C_s) + (\mu_MW_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2W_s^2V_{WW} \\ &\quad +(V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda\big\}. \end{aligned}$$

Because it is a necessary condition, the *first-order conditions* are

$$0 = u'(C_s) - V_W \quad \Rightarrow \quad V_W = u'(C_s), \tag{42}$$

$$\begin{aligned} 0 &= (\mu - r)W_sV_W + w_s\sigma^2W_s^2V_{WW} + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)(e^{\nu_1} - e^\kappa)W_sq\lambda \\ &\quad +V_W((1 + (e^{\nu_2} - 1)w_s)W_s)(1-q)(e^{\nu_2} - 1)W_s\lambda \end{aligned} \tag{43}$$

for any interior solution at any time $s = t \in [0, \infty)$.

Observe that (43) implies the optimal weight of the risky asset independent of the optimal consumption choice. We use this separation result throughout the paper by focusing on the consumption problem, given that income is generated by the uncertain yield of a (composite) asset, i.e., the optimal market portfolio (cf. also Merton, 1973).

The first-order condition (42) makes consumption a function of the state variable. Using the maximized Bellman equation for all $s = t \in [0, \infty)$,

$$\begin{aligned} \rho V(W_t) &= u(C(W_t)) + (\mu_M W_t - C(W_t))V_W + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WW} \\ &\quad + (V((1-\zeta_M)W_t)q + V((1-\zeta_M^{\text{o}})W_s)(1-q) - V(W_t))\lambda. \end{aligned}$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W &= (\mu_M V_W + (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + (V_W((1-\zeta_M)W_t)(1-\zeta_M)q + V_W((1-\zeta_M^{\text{o}})W_s)(1-\zeta_M^{\text{o}})(1-q) - V_W(W_t))\lambda. \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - \mu_M + \lambda)V_W &= (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + (V_W((1-\zeta_M)W_t)(1-\zeta_M)q + V_W((1-\zeta_M^{\text{o}})W_s)(1-\zeta_M^{\text{o}})(1-q))\lambda. \quad (44) \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W(W_t) &= (\mu_M W_t - C_t)\, V_{WW}dt + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW}dt + \sigma_M W_t V_{WW} dB_t \\ &\quad + (V_W((1-\zeta_M(t_-))W_{t-}) - V_W(W_{t-}))dN_t \\ &= \big((\rho - \mu_M + \lambda)V_W - \sigma_M^2 W_t V_{WW} - V_W((1-\zeta_M)W_t)(1-\zeta_M)q\lambda \\ &\quad - V_W((1-\zeta_M^{\text{o}})W_s)(1-\zeta_M^{\text{o}})(1-q)\lambda\big)dt \\ &\quad + \sigma_M W_t V_{WW} dB_t + (V_W((1-\zeta_M(t_-))W_{t-}) - V_W(W_{t-}))dN_t, \end{aligned}$$

where we inserted the costate from (44). As a final step we insert the first-order condition and obtain the Euler equation (8).

A.1.3 Proof of Proposition 2.1

The idea of this proof is to show that using an educated guess of the value function, the maximized Bellman equation and the first-order condition (7) are both fulfilled. For constant relative risk aversion, θ, the utility function has the form

$$u(C_t) = \mathbb{C}_1 \frac{C_t^{1-\theta}}{1-\theta} + \mathbb{C}_2, \quad \theta > 0,\ (\mathbb{C}_1, \mathbb{C}_2) \in \mathbb{R}_+ \times \mathbb{R}. \quad (45)$$

From (6), we obtain the maximized Bellman equation using the functional equation for consumption from the condition (7), i.e., $C(W_t) = \mathbb{C}_1^{1/\theta} V_W^{-1/\theta}$. We use the *educated guess*

$$\bar{V} = \mathbb{C}_0 \mathbb{C}_1 \frac{W_t^{1-\theta}}{1-\theta} + \mathbb{C}_2/\rho, \tag{46}$$

where $\bar{V}_W = \mathbb{C}_0\mathbb{C}_1 W_t^{-\theta}$ and $\bar{V}_{WW} = -\theta \mathbb{C}_0 \mathbb{C}_1 W_t^{-\theta-1}$, to solve the resulting equation. Note that optimal consumption is linear in wealth, $C(W_t) = \mathbb{C}_0^{-1/\theta} W_t$, and we arrive at

$$\begin{aligned}
\rho \mathbb{C}_0 \mathbb{C}_1 \frac{W_t^{1-\theta}}{1-\theta} + \mathbb{C}_2 &= \mathbb{C}_1 \frac{\mathbb{C}_0^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + \mathbb{C}_2 + \left(\mu_M W_t - \mathbb{C}_0^{-1/\theta} W_t \right) V_W + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WW} \\
&\quad + \left((1-\zeta_M)^{1-\theta} q + (1-\zeta_M^0)^{1-\theta}(1-q) - 1 \right) \mathbb{C}_0 \mathbb{C}_1 \frac{W_t^{1-\theta}}{1-\theta} \lambda.
\end{aligned}$$

Collecting terms gives

$$\begin{aligned}
\rho &= \mathbb{C}_0^{-1/\theta} + (1-\theta)\left(\mu_M - \mathbb{C}_0^{-1/\theta}\right) - (1-\theta)\tfrac{1}{2}\sigma_M^2 \theta \\
&\quad + \left((1-\zeta_M)^{1-\theta} q + (1-\zeta_M^0)^{1-\theta}(1-q) - 1 \right) \lambda \\
\Rightarrow \quad \mathbb{C}_0^{-1/\theta} &= \frac{\rho - (1-\theta)\mu_M + \lambda - (1-\zeta_M)^{1-\theta} q\lambda - (1-\zeta_M^0)^{1-\theta}(1-q)\lambda}{\theta} + (1-\theta)\tfrac{1}{2}\sigma_M^2 \\
&= \frac{\rho - (1-\theta)\mu_M + \lambda - (1-\zeta_M)^{1-\theta}\lambda}{\theta} + (1-\theta)\tfrac{1}{2}\sigma_M^2,
\end{aligned}$$

where the last equality used that in general equilibrium asset prices in (12) imply $\zeta_M = \zeta_M^0$. This proves that the guess (46) indeed is a solution, and by inserting the guess together with the constant, we obtain the consumption function. The effective risk aversion is obtained directly from the value function in (46).

A.2 A model of growth under uncertainty

A.2.1 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s, A_s) = \max_{C_s, H_s} \left\{ u(C_s, H_s) + \frac{1}{dt} E_s dV(W_s, A_s) \right\}.$$

Using Itô's formula yields

$$\begin{aligned}
dV &= V_W(dW_s - (e^{\nu_K} - 1) W_{s-} dN_t) + V_A dA_s + \tfrac{1}{2}\left(V_{AA}\sigma_A^2 A_s^2 + V_{WW}\sigma_K^2 W_s^2 \right) dt \\
&\quad + [V(W_s, A_s) - V(W_{s-}, A_{s-})] dN_t \\
&= ((r_s - \delta) W_s + H_s w_s^H - C_s) V_W dt + V_W \sigma_K W_s dZ_s + V_A \mu_A A_s dt + V_A \sigma_A A_s dB_s \\
&\quad + \tfrac{1}{2}\left(V_{AA}\sigma_A^2 A_s^2 + V_{WW}\sigma_K^2 W_s^2 \right) dt + [V(e^{\nu_K} W_{s-}, A_{s-}) - V(W_{s-}, A_{s-})] dN_t.
\end{aligned}$$

Using the property of stochastic integrals, we may write

$$\rho V(W_s, A_s) = \max_{C_s, H_s} \{u(C_s, H_s) + ((r_s - \delta)W_s + H_s w_s^H - C_s)V_W$$
$$+\tfrac{1}{2}\left(V_{AA}\sigma_A^2 A_s^2 + V_{WW}\sigma_K^2 W_s^2\right) + V_A\mu_A A_s + [V(e^{\nu_K}W_s, A_s) - V(W_s, A_s)]\lambda\}$$

for any $s \in [0, \infty)$. Because it is a necessary condition for optimality, we obtain the first-order conditions (25) and (26) which make optimal consumption and hours functions of the state variables, $C_t = C(W_t, A_t)$ and $H_t = H(W_t, A_t)$, respectively.

For the *evolution of the costate* we use the maximized Bellman equation

$$\rho V(W_t, A_t) = u(C(W_t, A_t), H(W_t, A_t)) + ((r_t - \delta)W_t + H(W_t, A_t)w_t^H - C(W_t, A_t))V_W$$
$$+V_A\mu_A A_t + \tfrac{1}{2}\left(V_{AA}\sigma_A^2 A_t^2 + V_{WW}\sigma_K^2 W_t^2\right) + [V(e^{\nu_K}W_t, A_t) - V(W_t, A_t)]\lambda, \qquad (47)$$

where $r_t = r(W_t, A_t)$ and $w_t^L = w(W_t, A_t)$ follow from the firm's optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,

$$\begin{aligned}\rho V_W \;&=\; \mu_A A_t V_{AW} + ((r_t - \delta)W_t + H_t w_t^H - C_t)V_{WW} + (r_t - \delta)V_W \\ &\quad +\tfrac{1}{2}\left(V_{WAA}\sigma_A^2 A_t^2 + V_{WWW}\sigma_K^2 W_t^2\right) + V_{WW}\sigma_K^2 W_t + [V_W(e^{\nu_K}W_t, A_t)e^{\nu_K} - V_W(W_t, A_t)]\lambda.\end{aligned}$$

Collecting terms we obtain

$$\begin{aligned}(\rho - (r_t - \delta) + \lambda)V_W \;&=\; V_{AW}\mu_A A_t + ((r_t - \delta)W_t + H_t w_t^H - C_t)V_{WW} \\ &\quad +\tfrac{1}{2}\left(V_{WAA}\sigma_A^2 A_t^2 + V_{WWW}\sigma_K^2 W_t^2\right) + \sigma_K^2 V_{WW} W_t + V_W(e^{\nu_K}W_t, A_t)e^{\nu_K}\lambda.\end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned}dV_W \;&=\; V_{AW}\mu_A A_t dt + V_{AW}\sigma_A A_t dB_t + \tfrac{1}{2}\left(V_{WAA}\sigma_A^2 A_t^2 + V_{WWW}\sigma_K^2 W_t^2\right)dt + V_{WW}\sigma_K W_t dZ_t \\ &\quad +((r_t - \delta)W_t + H_t w_t^H - C_t)V_{WW}dt + [V_W(W_t, A_t) - V_W(W_{t-}, A_{t-})]dN_t,\end{aligned}$$

where inserting yields

$$\begin{aligned}dV_W \;&=\; (\rho - (r_t - \delta) + \lambda)V_W dt - V_W(e^{\nu_K}W_t, A_t)e^{\nu_K}\lambda - \sigma_K^2 V_{WW} W_t dt + V_{AW} A_t \sigma_A dB_t \\ &\quad +V_{WW} W_t \sigma_K dZ_t + [V_W(e^{\nu_K}W_{t-}, A_{t-}) - V_W(W_{t-}, A_{t-})]dN_t,\end{aligned}$$

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (25) to obtain the Euler equation (28).

A.2.2 Proof of Proposition 3.1

The idea of this proof is to show that using an educated guess of the value function, the maximized Bellman equation (47) and the first-order condition (25) are both fulfilled. Since we assume that $\psi = 0$, we can neglect the intra-temporal first-order condition (26). The household will always supply total hours to production, $H = 1$, since there is no disutility from supplying labor. In what follows, we guess that the value function reads

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + f(A_t). \tag{48}$$

From (25), optimal consumption is a constant fraction of wealth,

$$C_t^{-\theta} = \mathbb{C}_1 W_t^{-\theta} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\theta} W_t.$$

Now use the maximized Bellman equation (47), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$, together with the transformation $K_t \equiv W_t$ (as the population size is normalized to one), and insert the solution candidate,

$$\begin{aligned} \rho \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (\alpha A_t W_t^{\alpha-1} W_t - \delta W_t + (1-\alpha) A_t W_t^{\alpha} - \mathbb{C}_1^{-1/\theta} W_t) \mathbb{C}_1 W_t^{-\theta} \\ &\quad -\tfrac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta} \sigma_K^2 - g(A_t) + (e^{(1-\theta)\nu_K} - 1) \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} \lambda, \end{aligned}$$

where we defined $g(A_t) \equiv \rho f(A_t) - f_A \mu_A A_t - \frac{1}{2} f_{AA} \sigma_A^2 A_t^2$. When imposing the condition $\alpha = \theta$ and $g(A_t) = \mathbb{C}_1 A_t$ it can be simplified to

$$\begin{aligned} (\rho - (e^{(1-\theta)\nu_K} - 1)\lambda) \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + g(A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (A_t W_t^{\alpha-\theta} - \delta W_t^{1-\theta} - \mathbb{C}_1^{-1/\theta} W_t^{1-\theta}) \mathbb{C}_1 \\ &\quad -\tfrac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta} \sigma_K^2 \\ \Leftrightarrow \; (\rho - (e^{(1-\theta)\nu_K} - 1)\lambda) W_t^{1-\theta} &= \theta \mathbb{C}_1^{-1/\theta} W_t^{1-\theta} - (1-\theta)\delta W_t^{1-\theta} - \tfrac{1}{2}\theta(1-\theta) W_t^{1-\theta} \sigma_K^2, \end{aligned}$$

which implies that $\mathbb{C}_1^{-1/\theta} = \left(\rho - (e^{(1-\theta)\nu_K} - 1)\lambda + (1-\theta)\delta + \frac{1}{2}\theta(1-\theta)\sigma_K^2\right)/\theta$. This proves that the guess (48) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal consumption function.

A.2.3 Proof of Proposition 3.3

The idea of this proof follows Section A.2.2. An educated guess of the value function is

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta}. \tag{49}$$

From the first-order conditions (25) and (26), we obtain

$$\begin{aligned} C_t^{-\theta} (1-H_t)^{(1-\theta)\psi} &= \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta}, \\ \psi C_t^{1-\theta} (1-H_t)^{(1-\theta)\psi-1} &= w_t^H \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \quad \Rightarrow \quad \psi C_t/(1-H_t) = (1-\alpha) A_t W_t^{\alpha} H_t^{-\alpha}. \end{aligned}$$

Suppose that optimal hours are constant, $H_t = H$, then optimal consumption becomes a constant fraction of income,

$$C_t = (1-s)A_t W_t^{\alpha} H^{1-\alpha}, \quad 1-s \equiv (1-\alpha)\frac{1-H}{\psi H}, \quad \psi \neq 0.$$

Inserting everything into (47) and collecting terms gives

$$\begin{aligned}
&\left(\rho + (1-\alpha\theta)\delta + \left(\theta\mu_A - \tfrac{1}{2}\left(\theta(1+\theta)\sigma_A^2 - \alpha\theta(1-\alpha\theta)\sigma_K^2\right)\right) - (e^{\nu_K(1-\alpha\theta)}-1)\lambda\right)\frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta}A_t^{-\theta} \\
&= \left((1-s)^{1-\theta}H^{(1-\theta)(1-\alpha)}(1-H)^{(1-\theta)\psi} + \left(H^{1-\alpha} - (1-s)H^{1-\alpha}\right)(1-\theta)\mathbb{C}_1\right)\frac{A_t^{1-\theta}W_t^{\alpha-\alpha\theta}}{1-\theta}.
\end{aligned}$$

Hence, for $\rho = \bar{\rho}$ and

$$\mathbb{C}_1 = -\frac{(1-s)^{1-\theta}H^{(1-\alpha)(1-\theta)}(1-H)^{(1-\theta)\psi}}{(1-\theta)H^{1-\alpha} - (1-\theta)(1-s)H^{1-\alpha}},$$

the constant savings rate is indeed the optimal solution. The optimal hours can be obtained from the first-order condition for consumption

$$\begin{aligned}
C_t(1-H)^{-\frac{1-\theta}{\theta}\psi} &= \mathbb{C}_1^{-1/\theta}W_t^{\alpha}A_t \\
\Leftrightarrow \quad \frac{1-\alpha}{\psi}H^{-\alpha}(1-H)^{1-\frac{1-\theta}{\theta}\psi} &= \mathbb{C}_1^{-1/\theta}.
\end{aligned}$$

Inserting the condition for $\mathbb{C}_1$, we obtain

$$\begin{aligned}
\left(\frac{1-\alpha}{\psi}\right)^{-\theta}H^{\alpha\theta}(1-H)^{-\theta+(1-\theta)\psi} &= -\frac{(1-s)^{1-\theta}H^{(1-\alpha)(1-\theta)}(1-H)^{(1-\theta)\psi}}{(1-\theta)H^{1-\alpha} - (1-\theta)(1-s)H^{1-\alpha}} \\
\Leftrightarrow \quad \frac{\psi}{1-\alpha} &= -\frac{1-H}{(1-\theta)H - (1-\theta)(1-\alpha)(1-H)/\psi}.
\end{aligned}$$

Collecting terms yields

$$\begin{aligned}
\psi &= -\frac{(1-\alpha)(1-H)}{(1-\theta)H - (1-\theta)(1-\alpha)(1-H)/\psi} \\
\Leftrightarrow \quad -\psi(1-\theta)H &= \theta(1-\alpha)(1-H) \\
\Leftrightarrow \quad H &= \frac{\theta(1-\alpha)}{\theta(1-\alpha) - \psi(1-\theta)}
\end{aligned}$$

which are admissible solutions if and only if $0 < H < 1$, which holds for $\theta > 1$.

A.2.4 Obtaining the dynamic equilibrium system

We employ both first-order conditions (25) and (26) to substitute the costate V_W and obtain Euler equations for both optimal consumption and hours. For consumption it gives (28), which for our simplifying assumptions $\sigma_K = \sigma_A = \mu_A = 0$ reduces to

$$
\begin{aligned}
du_C &= (\rho - (r_t - \delta) + \lambda) u_C dt - u_C(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t)) e^{\nu_K} \lambda dt \\
&\quad + \left(u_C(C(e^{\nu_K} W_{t-}), H(e^{\nu_K} W_{t-})) - u_C(C_{t-}, H_{t-})\right) dN_t \\
\Leftrightarrow \; dC_t &= \frac{u_C}{u_{CC}} (\rho - (r_t - \delta) + \lambda) dt - \frac{u_C}{u_{CC}} \frac{u_C(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t))}{u_C(C(W_t), H(W_t))} e^{\nu_K} \lambda dt \\
&\quad - \frac{u_{CH}}{u_{CC}} \left(dH_t - (H(e^{\nu_K} W_{t-}) - H(W_{t-})) dN_t\right) + \left(C(e^{\nu_K} W_{t-}) - C(W_{t-})\right) dN_t. \quad (50)
\end{aligned}
$$

Similarly, we use the first-order condition for hours (26), and replace V_W by $-u_H / w_t^H$,

$$
\begin{aligned}
d(u_H / Y_H) &= (\rho - (r_t - \delta) + \lambda) u_H / Y_H dt \\
&\quad - u_H(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t)) / Y_H(e^{\nu_K} W_t, H(e^{\nu_K} W_t)) e^{\nu_K} \lambda dt \\
&\quad + \left[\frac{u_H(C(e^{\nu_K} W_{t-}), H(e^{\nu_K} W_{t-}))}{Y_H(e^{\nu_K} W_{t-}, H(e^{\nu_K} W_{t-}))} - \frac{u_H(C(W_{t-}), H(W_{t-}))}{Y_H(W_{t-}, H(W_{t-}))} \right] dN_t \\
\Leftrightarrow \; du_H &= (\rho - (r_t - \delta) + \lambda) u_H dt - u_H(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t)) \frac{Y_H(W_t, H_t) e^{\nu_K} \lambda}{Y_H(e^{\nu_K} W_t, H(e^{\nu_K} W_t))} dt \\
&\quad + u_H / Y_H (dY_H - (Y_H(W_t, H_t) - Y_H(W_{t-}, H_{t-})) dN_t) \\
&\quad + (u_H(C_t, H_t) - u_H(C_{t-}, H_{t-})) dN_t,
\end{aligned}
$$

in which we may use $Y_H = Y_H(W_t, H_t)$. Thus, we get

$$
\begin{aligned}
dY_H &= Y_{HH}(dH_t - (H_t - H_{t-}) dN_t) + Y_{HK}(dW_t - (W_t - W_{t-}) dN_t) \\
&\quad + (Y_H(W_t, H_t) - Y_H(W_{t-}, H_{t-})) dN_t.
\end{aligned}
$$

Use the inverse function theorem to obtain

$$
\begin{aligned}
dH_t &= \frac{u_H}{u_{HH}} (\rho - (r_t - \delta) + \lambda) dt - \frac{u_H}{u_{HH}} \frac{u_H(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t))}{u_H(C(W_t), H(W_t))} \frac{Y_H(W_t, H_t) e^{\nu_K} \lambda}{Y_H(e^{\nu_K} W_t, H(e^{\nu_K} W_t))} dt \\
&\quad - \frac{u_{HC}}{u_{HH}} (dC_t - (C_t - C_{t-}) dN_t) + \frac{u_H}{u_{HH} Y_H} (dY_H - (Y_H(W_t, H_t) - Y_H(W_{t-}, H_{t-})) dN_t) \\
&\quad + (H(W_t) - H(W_{t-})) dN_t.
\end{aligned}
$$

Finally we insert the differentials for consumption and for the wage rate. After some tedious algebra and by collecting partial derivatives, $\bar{u} \equiv u_{CH} u_{HC} - u_{CC} u_{HH}$, we arrive at

$$
\begin{aligned}
dH_t &= \frac{u_{HC} u_C - u_{CC} u_H}{Y_{HH} / Y_H u_H u_{CC} + \bar{u}} (\rho - (r_t - \delta) + \lambda) dt \\
&\quad - \frac{u_{CC} u_H Y_{HK} / Y_H}{Y_{HH} / Y_H u_H u_{CC} + \bar{u}} ((r_t - \delta) W_t + H_t w_t^H - C_t) dt \\
&\quad + \frac{u_{CC} u_H}{Y_{HH} / Y_H u_H u_{CC} + \bar{u}} \frac{u_H(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t))}{u_H(C(W_t), H(W_t))} \frac{Y_H(W_t, H_t)}{Y_H(e^{\nu_K} W_t, H(e^{\nu_K} W_t))} e^{\nu_K} \lambda dt \\
&\quad - \frac{u_{HC} u_C}{Y_{HH} / Y_H u_H u_{CC} + \bar{u}} \frac{u_C(C(e^{\nu_K} W_t), H(e^{\nu_K} W_t))}{u_C(C(W_t), H(W_t))} e^{\nu_K} \lambda dt + (H_t - H_{t-}) dN_t. \quad (51)
\end{aligned}
$$

In summary, the system of (50), (51), together with the budget constraint,

$$dW_t = ((r_t - \delta)W_t + H_t w_t^H - C_t)dt - (1 - e^{\nu_K})W_{t-}dN_t, \tag{52}$$

define the reduced form description of the dynamic equilibrium system.

A.2.5 The dynamic system for Cobb-Douglas production and CRRA utility

In this section we derive the equilibrium system for the case of Cobb-Douglas production, $Y_t = AK_t^\alpha H_t^{1-\alpha}$ and CRRA utility as in (30). We may use the partial derivatives

$$u_C = C_t^{-\theta}(1 - H_t)^{(1-\theta)\psi}, \quad u_H = -\psi C_t^{1-\theta}(1 - H_t)^{(1-\theta)\psi-1}, \quad u_{CC} = -\theta C_t^{-\theta-1}(1 - H_t)^{(1-\theta)\psi},$$
$$u_{HH} = ((1-\theta)\psi - 1)\psi C_t^{1-\theta}(1 - H_t)^{(1-\theta)\psi-2}, \quad u_{HC} = -(1-\theta)\psi C_t^{-\theta}(1 - H_t)^{(1-\theta)\psi-1},$$
$$Y_H = (1-\alpha)AK_t^\alpha H_t^{-\alpha}, \quad Y_{HH} = -\alpha(1-\alpha)AK_t^\alpha H_t^{-\alpha-1}, \quad Y_{HK} = \alpha(1-\alpha)AK_t^{\alpha-1}H_t^{-\alpha},$$
$$Y_{HH}/Y_H = -\alpha H_t^{-1}, \quad Y_{HK}/Y_H = \alpha K_t^{-1},$$

to obtain our dynamic system in three variables C_t, H_t and W_t. As from Section 3.3.3, we may neglect the first equation since for any interior solution $C_t = C(H(W_t), W_t)$. We get

$$\begin{aligned}
dH_t \;=\; & \frac{-(1-\theta)\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1} - \theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}}{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}Y_{HH}/Y_H + \bar{u}}(\rho - (r_t - \delta) + \lambda)dt \\
& - \frac{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}Y_{HK}/Y_H}{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}Y_{HH}/Y_H + \bar{u}}((r_t - \delta)W_t + H_t w_t^H - C_t)dt \\
& + \frac{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}}{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}Y_{HH}/Y_H + \bar{u}} \frac{u_H(C(e^{\nu_K}W_t), H(e^{\nu_K}W_t))}{u_H(C(W_t), H(W_t))} \\
& \times \frac{Y_H(W_t, H_t)}{Y_H(e^{\nu_K}W_t, H(e^{\nu_K}W_t))} e^{\nu_K}\lambda dt \\
& - \frac{-(1-\theta)\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}}{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}Y_{HH}/Y_H + \bar{u}} \frac{u_C(C(e^{\nu_K}W_t), H(e^{\nu_K}W_t))}{u_C(C(W_t), H(W_t))} e^{\nu_K}\lambda dt \\
& + (H_t - H_{t-})dN_t,
\end{aligned}$$

where

$$\begin{aligned}
\bar{u} \;&=\; (1-\theta)^2\psi^2 C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} + (\theta\psi^2 - \theta^2\psi^2 - \psi\theta)C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} \\
&=\; \left((1-\theta)^2\psi^2 + \theta\psi^2 - \theta^2\psi^2 - \psi\theta\right) C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} \\
&=\; \psi\left(\psi - \theta\psi - \theta\right) C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2}.
\end{aligned}$$

Using the equilibrium condition (38), the jump terms are given by

$$\begin{aligned}
\frac{u_H(C(e^{\nu_K}W_t),H(e^{\nu_K}W_t))}{u_H(C(W_t),H(W_t))} &= \frac{C(e^{\nu_K}W_t)^{1-\theta}\left(C(e^{\nu_K}W_t)H(e^{\nu_K}W_t)^{\alpha}e^{-\alpha\nu_K}W_t^{-\alpha}\right)^{(1-\theta)\psi-1}}{C(W_t)^{1-\theta}\left(C(W_t)H(W_t)^{\alpha}W_t^{-\alpha}\right)^{(1-\theta)\psi-1}} \\
&= \tilde{C}(W_t)^{-\theta+(1-\theta)\psi}\tilde{H}(W_t)^{(1-\theta)\psi\alpha-\alpha}e^{-(1-\theta)\alpha\nu_K\psi+\alpha\nu_K}, \\
\frac{u_C(C(e^{\nu_K}W_t),H(e^{\nu_K}W_t))}{u_C(C(W_t),H(W_t))} &= \frac{C(e^{\nu_K}W_t)^{-\theta}\left(C(e^{\nu_K}W_t)H(e^{\nu_K}W_t)^{\alpha}e^{-\alpha\nu_K}W_t^{-\alpha}\right)^{(1-\theta)\psi}}{C(W_t)^{-\theta}\left(C(W_t)H(W_t)^{\alpha}W_t^{-\alpha}\right)^{(1-\theta)\psi}} \\
&= \tilde{C}(W_t)^{-\theta+(1-\theta)\psi}\tilde{H}(W_t)^{(1-\theta)\psi\alpha}e^{-(1-\theta)\psi\alpha\nu_K}, \\
\frac{Y_H(W_t,H_t)}{Y_H(e^{\nu_K}W_t,H(e^{\nu_K}W_t))} &= \frac{H(W_t)^{-\alpha}}{e^{\alpha\nu_K}H(e^{\nu_K}W_t)^{-\alpha}} = \tilde{H}(W_t)^{\alpha}e^{-\alpha\nu_K}.
\end{aligned}$$

Collecting terms we may obtain

$$\begin{aligned}
dH_t &= \frac{-\left((1-\theta)\psi+\theta\psi\right)(\rho-r_t+\delta+\lambda)-\theta\psi Y_{HK}/Y_H((r_t-\delta)W_t+H_tw_t^H-C_t)}{\theta\psi Y_{HH}/Y_H+\psi\left(\psi-\theta\psi-\theta\right)(1-H_t)^{-1}}dt \\
&\quad+\frac{\psi\tilde{C}(W_t)^{-\theta+(1-\theta)\psi}\tilde{H}(W_t)^{(1-\theta)\psi\alpha}e^{\nu_k-(1-\theta)\psi\alpha\nu_K}\lambda}{\theta\psi Y_{HH}/Y_H+\psi\left(\psi-\theta\psi-\theta\right)(1-H_t)^{-1}}dt+(H_t-H_{t-})dN_t.
\end{aligned}$$

Inserting the remaining partial derivatives gives

$$\begin{aligned}
dH_t &= \frac{\rho-(1-\theta)r_t+\delta+\lambda-\theta\alpha\delta-\theta\alpha C_t/W_t}{\alpha\theta H_t^{-1}-\left(\psi-\theta\psi-\theta\right)(1-H_t)^{-1}}dt \\
&\quad+\frac{-\tilde{C}(W_t)^{-\theta+(1-\theta)\psi}\tilde{H}(W_t)^{(1-\theta)\psi\alpha}e^{\nu_k-(1-\theta)\psi\alpha\nu_K}\lambda}{\alpha\theta H_t^{-1}-\left(\psi-\theta\psi-\theta\right)(1-H_t)^{-1}}dt+(H_t-H_{t-})dN_t,
\end{aligned}$$

which is the evolution of optimal hours in the reduced form description in the main text.

A.3 Tables and Figures

Table A.1: Calibrated model and the risk premium (endowment economy)

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
						Parameters		
		No disasters	Baseline	Low θ	High λ	Low q	Low μ_A	Low ρ
θ	(coef. of relative risk aversion)	4	**4**	**3**	4	4	4	4
σ_A	(s.d. of growth rate, no disasters)	0.02	**0.02**	0.02	0.02	0.02	0.02	0.02
ρ	(rate of time preference)	0.03	**0.03**	0.03	0.03	0.03	0.03	**0.02**
μ_A	(growth rate, deterministic part)	0.025	**0.025**	0.025	0.025	0.025	**0.020**	0.025
λ	(disaster probability)	0	**0.017**	0.017	**0.025**	0.017	0.017	0.017
q	(default probability in disaster)	0	**0.4**	0.4	0.4	**0.3**	0.4	0.4
$1 - e^{\nu_A}$	(size of disaster)	0	**0.4**	0.4	0.4	0.4	0.4	0.4
$1 - e^{\kappa}$	(size of default)	0	**0.4**	0.4	0.4	0.4	0.4	0.4
						Variables		
Default risk		0	**0.021**	0.012	0.03	0.016	0.021	0.021
Disaster risk		0	**0.031**	0.019	0.046	0.036	0.031	0.031
Diffusion risk		0.002	**0.002**	0.001	0.002	0.002	0.002	0.002
Risk premium		0.002	**0.054**	0.032	0.078	0.054	0.054	0.054
Expected market rate		0.128	**0.06**	0.067	0.028	0.06	0.04	0.05
Expected bill rate		0.126	**0.031**	0.051	-0.013	0.026	0.011	0.021
Market premium		0.002	**0.029**	0.016	0.041	0.033	0.029	0.029
Expected market rate, conditional on no disaster		0.128	**0.066**	0.074	0.038	0.066	0.046	0.056
Face bill rate		0.126	**0.034**	0.054	-0.009	0.028	0.014	0.024
Market premium, conditional on no disaster		0.002	**0.033**	0.02	0.047	0.038	0.033	0.033
Sharpe ratio, conditional on no disaster		0.08	**1.641**	0.996	2.366	1.901	1.641	1.641
Expected growth rate		0.025	**0.016**	0.016	0.012	0.016	0.011	0.016
Expected growth rate, conditional on no disaster		0.025	**0.025**	0.025	0.025	0.025	0.02	0.025

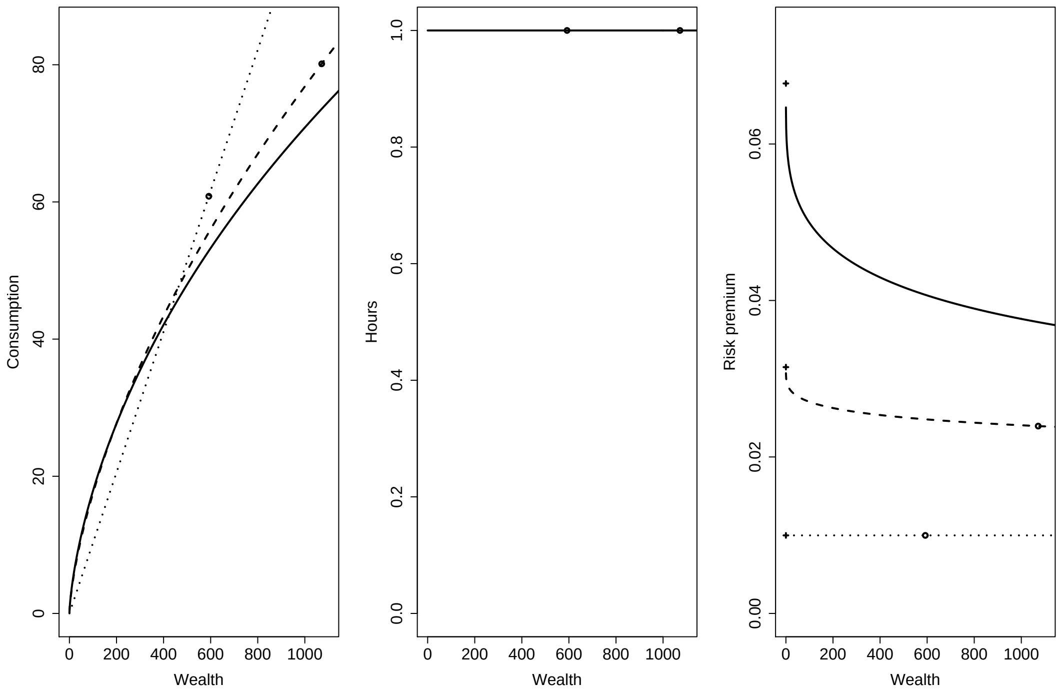
Table A.2: Calibrated model and the risk premium (endowment economy)

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
						Parameters		
		No default	Baseline	High σ_A	High λ	High q	Low $1-e^{\nu_A}$	Low $1-e^{\kappa}$
θ	(coef. of relative risk aversion)	4	**4**	4	4	4	4	4
σ_A	(s.d. of growth rate, no disasters)	0.02	**0.02**	**0.05**	0.02	0.02	0.02	0.02
ρ	(rate of time preference)	0.03	**0.03**	0.03	0.03	0.03	0.03	0.03
μ_A	(growth rate, deterministic part)	0.025	**0.025**	0.025	0.025	0.025	0.025	0.025
λ	(disaster probability)	0.017	**0.017**	0.017	**0.2**	0.017	0.017	0.017
q	(default probability in disaster)	**0**	**0.4**	0.4	0.4	**1**	0.4	0.4
$1-e^{\nu_A}$	(size of disaster)	0.4	**0.4**	0.4	**0.034**	0.4	**0.2**	0.4
$1-e^{\kappa}$	(size of default)	0.4	**0.4**	0.4	**0.034**	0.4	0.4	**0.2**
						Variables		
Default risk		0	**0.021**	0.021	0.003	0.052	0.007	0.01
Disaster risk		0.052	**0.031**	0.031	0.004	0	0.002	0.042
Diffusion risk		0.002	**0.002**	0.01	0.002	0.002	0.002	0.002
Risk premium		0.054	**0.054**	0.062	0.009	0.054	0.01	0.054
Expected market rate		0.06	**0.06**	0.047	0.102	0.06	0.108	0.06
Expected bill rate		0.013	**0.031**	0.01	0.099	0.058	0.106	0.022
Market premium		0.047	**0.029**	0.037	0.002	0.002	0.003	0.038
Expected market rate, conditional on no disaster		0.066	**0.066**	0.054	0.108	0.066	0.112	0.066
Face bill rate		0.013	**0.034**	0.013	0.102	0.065	0.108	0.023
Market premium, conditional conditional on no disaster		0.054	**0.033**	0.041	0.006	0.002	0.003	0.043
Sharpe ratio, conditional on no disaster		2.681	**1.641**	0.824	0.292	0.08	0.162	2.161
Expected growth rate		0.016	**0.016**	0.015	0.019	0.016	0.021	0.016
Expected growth rate, conditional on no disaster		0.025	**0.025**	0.024	0.025	0.025	0.025	0.025

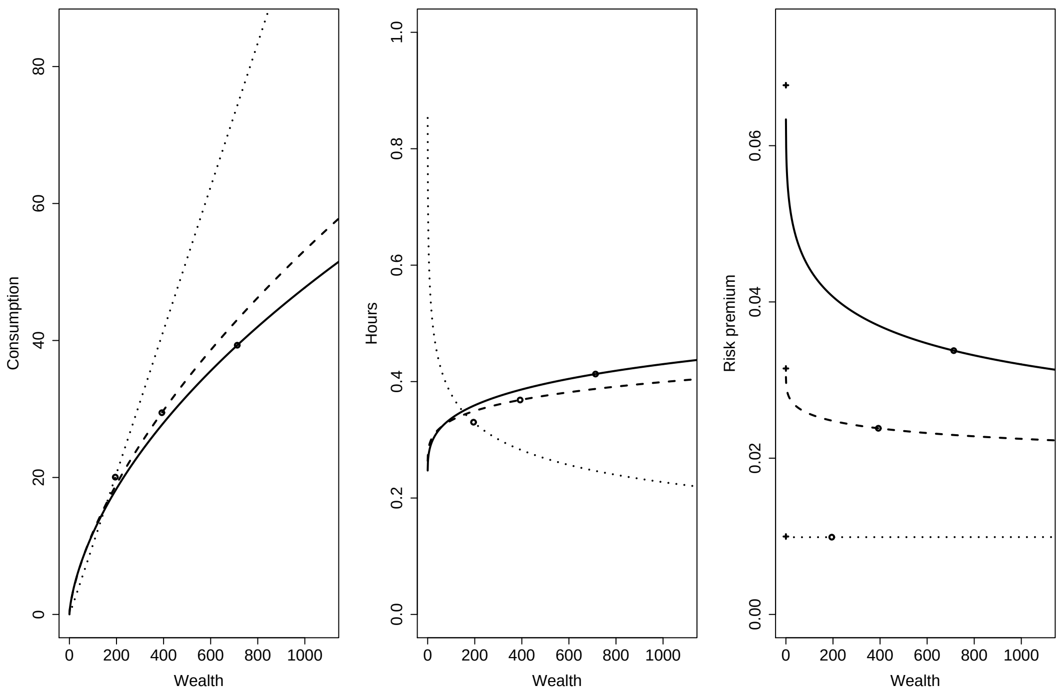
Table A.3: Calibrated model and the risk premium (production economy)

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
						Parameters		
		No disasters	Baseline	High θ	Low α	Low δ	High λ	High $\|\nu_K\|$
θ	(coef. of relative risk aversion)	4	**4**	**6**	4	4	4	4
α	(output elasticity of capital)	0.75	**0.75**	0.75	**0.33**	0.75	0.75	0.75
δ	(capital depreciation, deterministic part)	0.1	**0.1**	0.1	0.1	**0.05**	0.1	0.1
ρ	(rate of time preference)	0.05	**0.05**	0.05	0.05	0.05	0.05	0.05
σ_K	(s.d. of stochastic depreciation, no disasters)	0	**0**	0	0	0	0	0
σ_A	(s.d. of TFP growth)	0	**0**	0	0	0	0	0
μ_A	(growth rate TFP, deterministic part)	0	**0**	0	0	0	0	0
λ	(disaster probability)	**0**	**0.017**	0.017	0.017	0.017	**0.02**	0.017
$1 - e^{\nu_K}$	(size of disaster)	0	**0.4**	0.4	0.4	0.4	0.4	**0.5**
						Variables		
Implied knife-edge value $\bar{\rho}$		0.200	**0.230**	0.435	0.035	0.130	0.236	0.251
Risk premium								
steady state		0	**0.024**	0.034	0.014	0.027	0.028	0.045
zero wealth (left limit)		0	**0.032**	0.068	0.013	0.032	0.037	0.068
Market rate, steady state (gross)		0.150	**0.131**	0.116	0.147	0.077	0.128	0.122
Bill rate, steady state (gross)		0.150	**0.107**	0.081	0.133	0.051	0.101	0.078
Market rate, steady state (net)		0.050	**0.031**	0.016	0.047	0.027	0.028	0.022
Bill rate, steady state (net)		0.050	**0.007**	-0.019	0.033	0.001	0.001	-0.022

Figure A.1: Risk premia in a production economy

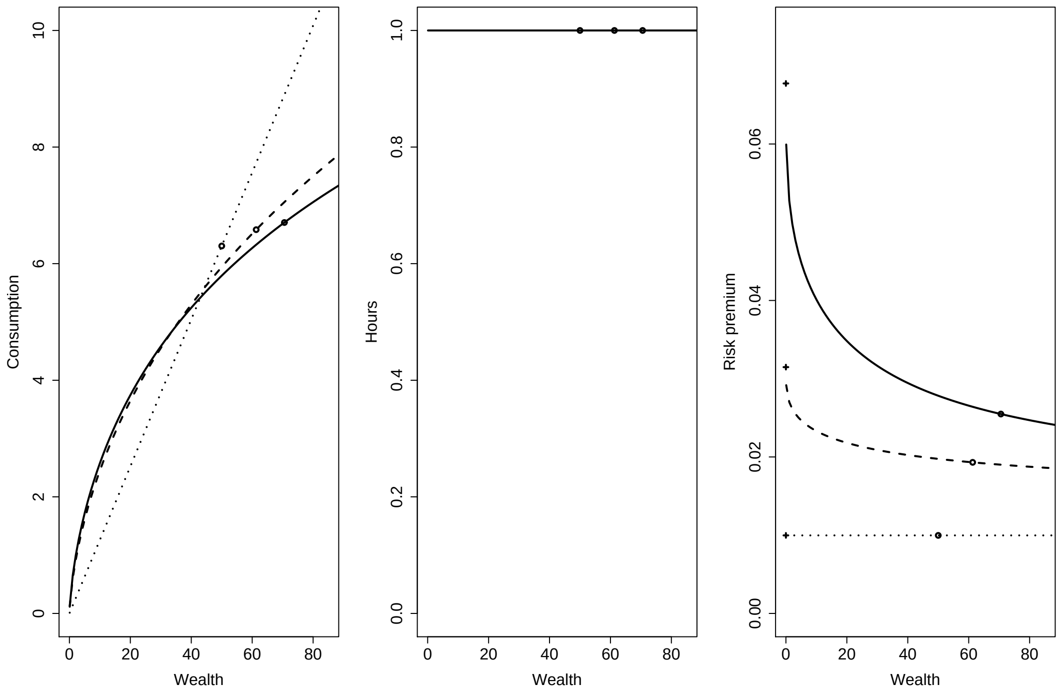


Notes: These figures illustrate optimal consumption (left panel), optimal hours (middle panel) and the risk premium (right panel) as functions of wealth (∘ are steady-state values) for different levels of relative risk aversion and $\sigma_K = \sigma_A = \mu_A = 0$; other parameters $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu_K}, \psi) = (0.05, 0.75, \cdot, 0.1, 0.017, 0.4, 0)$; $\theta = 0.75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).

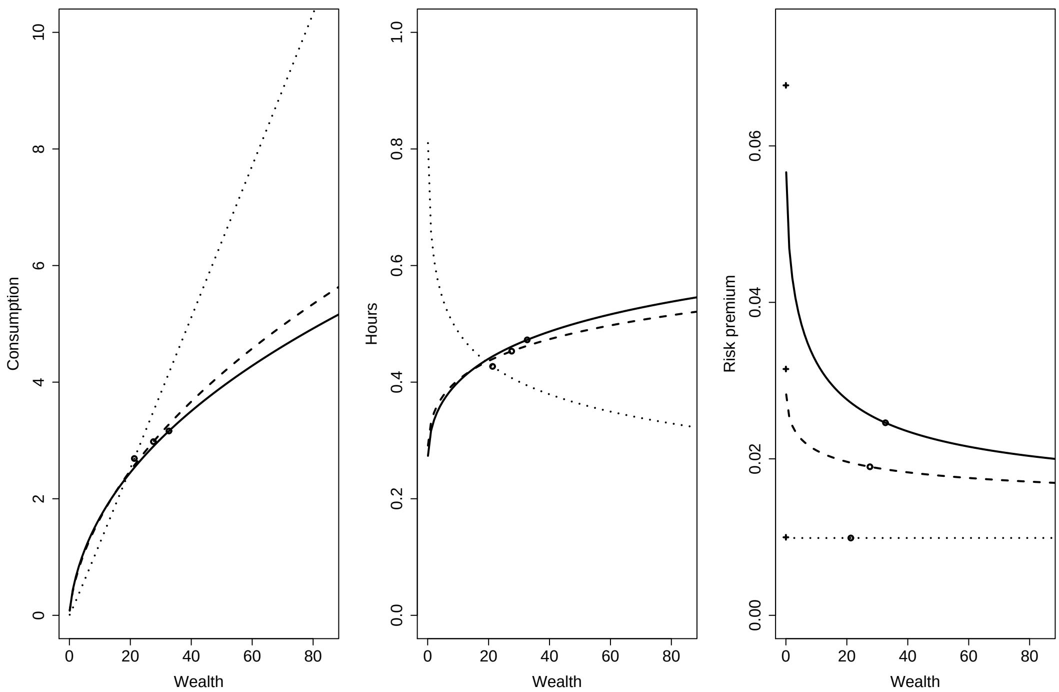


Notes: These figures illustrate optimal consumption (left panel), optimal hours (middle panel) and the risk premium (right panel) as functions of wealth (∘ are steady-state values) for different levels of relative risk aversion and $\sigma_K = \sigma_A = \mu_A = 0$; other parameters $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu_K}, \psi) = (0.05, 0.75, \cdot, 0.1, 0.017, 0.4, 1)$; $\theta = 0.75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).

Figure A.2: Risk premia in a production economy



Notes: These figures illustrate optimal consumption (left panel), optimal hours (middle panel) and the risk premium (right panel) as functions of wealth (∘ are steady-state values) for different levels of relative risk aversion and $\sigma_K = \sigma_A = \mu_A = 0$; other parameters $(\rho, \alpha, \theta, \delta, \lambda, 1-e^{\nu_K}, \psi) = (0.03, 0.75, \cdot, 0.25, 0.017, 0.4, 0)$; $\theta = 0.75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).



Notes: These figures illustrate optimal consumption (left panel), optimal hours (middle panel) and the risk premium (right panel) as functions of wealth (∘ are steady-state values) for different levels of relative risk aversion and $\sigma_K = \sigma_A = \mu_A = 0$; other parameters $(\rho, \alpha, \theta, \delta, \lambda, 1-e^{\nu_K}, \psi) = (0.03, 0.75, \cdot, 0.25, 0.017, 0.4, 1)$; $\theta = 0.75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).